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ACTION OF THE PLANETS ON THE MOON

WASHINGTON
BUREAU OF EQUIPMENT, NAVY DEPARTMENT
1894

III.

THEORY

OF THE

INEQUALITIES IN THE MOTION OF THE MOON

PRODUCED BY

THE ACTION OF THE PLANETS

BY

SIMON NEWCOMB.

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PREFACE.

The fact that the observed motion of the Moon is apparently affected by inequalities of long period not yet satisfactorily accounted for is well known to all who have interested themselves in the subject. The study of these inequalities was undertaken by the author about the year 1870. The discussion of such observations as seemed best fitted to throw light on the subject was made at the U. S. Naval Observatory, and published in the year 1878, as an appendix to the Washington Observations for 1875. The result of that discussion was that the inequalities in question could not be represented by any existing theory, and might, therefore, be attributed to changes in the Earth's axial rotation, if it could be shown by other independent observations that such changes really existed. In the absence of such independent proof an attempt was made to represent the outstanding differences between theory and observation by an empirical formula; and the curious result was reached that by a quite simple change in what is now known as **Hansen's** first inequality of long period a fairly good agreement with observation could be brought about.

The changes indicated by this formula were made in the mean longitude of **Hansen's** lunar tables, and the observations up to the year 1863 show that the corrected motion agrees remarkably well with the fact. The simple way in which this agreement was brought about would seem to point very strongly to the theoretical correctness of the proposed change. Yet the most rigid examination into the subject has failed to show the possibility of any error in **Hansen's** result.

Meanwhile an independent test of the constancy of the Earth's axis of rotation has been made through the discussion of the observed transits of Mercury, published in the first volume of the present series. The results of this test were inconclusive, in that, while it seemed proved very clearly that the apparent inequalities in the motion of the Moon were not wholly due to changes in the Earth's rotation, yet such changes were indicated in a somewhat inconclusive way, having a certain general analogy to those previously inferred from the motion of the Moon, though much smaller in amount, and therefore not very certain. What is most remarkable in this connection is that a study of the motions of the Moon, Mercury, and the first satellite of Jupiter all agree in indicating that, during some years previous to 1863, there was an increase in the speed of the rotation of the Earth which, however, rapidly subsided about that epoch. Added importance was thus given to the theoretical side of the question.

Among the problems of celestial mechanics which admit of rigorous treatment, it is well understood that that of the action of the planets on the Moon is the most difficult which presents itself. It was therefore necessary, in commencing the work, to devise a method by which it was supposed these difficulties could best be surmounted, and especially, by which every possible inequality could be detected and investigated. It occurred to the author that a more rigorous and general treatment than that hitherto adopted would be to consider the arbitrary constants which enter into the solution of the problem of three bodies, the Sun, Moon, and Earth, as simultaneously varied by the action of a fourth body, a planet, using LA GRANGE's method of the variation of elements. The determination of the LA GRANGIAN coefficients necessary in the theory proved to be a work of more simplicity and elegance than would have been supposed. Accordingly he took advantage of a visit to Paris in April, 1871, to read, before the French Academy, a preliminary chapter on the subject, which was soon afterward published in *Liouville's Journal de Mathématiques Pures et Appliquées.* In the course of the next two years the results of DELAUNAY's great work were utilized for the purpose of forming the actual values of the expressions which entered into the theory, and the greater part of the numerical developments pertaining to the action of the individual planets was also effected. The computation of the inequalities of long period produced by the action of Venus and Mars was also supposed to be nearly completed.

At this stage of the work a more careful investigation was made of its foundation, leading to the discovery that the author had wrongly measured the importance of quantities in the theory of three bodies by the numerical value of the perturbations to which they gave rise. The motion of the center of gravity of the Earth and Moon around the Sun, as determined by the action of the Sun itself, deviates from an invariable ellipse by quantities of no practical importance in ordinary astronomy. It was, therefore, supposed at the outset that the elements of the orbit of the Moon need not be regarded as contained in this motion. An examination of the problem showed, however, that this conclusion was ill-founded, and that, so far as mere order of magnitude went, the consideration of the lunar elements in this motion was as necessary as in the motion of the Moon itself. It therefore seemed to be necessary to form expressions for the solar elements in terms of the lunar elements, a work which, if completely executed, might be very laborious. Under these circumstances the work was laid aside until the author should find the leisure necessary to carry through the required investigations. Attempts were made from time to time to overcome or evade the difficulty, but none of them were brought to an entirely satisfactory conclusion. Indeed, the author is quite prepared for the conclusion that the rule found to hold in the theories of the planets, that the more general and abstractly rigorous a method of considering perturbations is, the less it is adapted to their actual numerical computation, may hold true in the present case. He has, however, at length deemed it best to publish the work, notwithstanding its lack of completeness, for the following reasons:

1. Its analogy with some of the modern theories of celestial mechanics, and its possible value as illustrating those theories, especially in the form developed by POINCARÉ in his recent treatise *Les méthodes nouvelles de la Mécanique céleste.* The path on which it proceeds to extend the solution of the problem of three bodies to that of four may perhaps lead to the method by which the most general solution of the equations of planetary motion is hereafter to be investigated.

2. The numerical developments for the action of the planets, from Mercury to Jupiter, inclusive, are equally necessary in applying any other method. Indeed, one circumstance which led to the question of publication being considered at the present moment, was the remark that these numerical developments might perhaps have been of use to RADAU in his recent elaborate researches on the subject.

3. Many of the analytic developments of the present method will be found closely analogous to, or identical with, those of other methods, and may thus serve as a control on the latter, or as a basis for their immediate application.

4. We have also to consider the usefulness of even an abstract method of considering the theory in its most general form, in that a more luminous view of the scope and principles of other methods may be obtained.

5. Finally, the author is not without hope that some way of avoiding or overcoming the difficulties he has encountered will be devised by an abler investigator; or, at least, that such an investigator will find the work useful. In any case the problem of applying the method is now merely that of completing the analytic theory of the motion of three bodies.

The first part of the work, relating to the general dynamical theory of the subject, was reconstructed at various times between the years 1873 and 1888, with a view of putting it in such a shape as would best admit of farther development and wider application. This remark applies especially to §§ 2 to 4, which, though first in order, are practically the last that have been prepared. All the rest of the work is now published as originally written in the years 1871–1873, except ordinary revision of details. Being once shaped into a logical and continuous whole, it was scarcely possible to make any extended changes or additions, except by a process which would have been equivalent to a complete reconstruction. The author was the more willing to refrain from such a course because he might then be more readily excused if it should appear that he had not duly considered the advances made within the last twenty years in the theory of the subject. By this course he has made plain an oversight in his conclusions as to the possibility of inequalities being produced by Jupiter, whereby what is now known as the JOVIAN evection was left to be discovered by Mr. NEVILL.

It may not be amiss in this connection to call attention to the fact that the problem has lost none of its importance during the time which has elapsed. There are three possible sources to which may be attributed the apparent inequalities in question:

(α) Defects in the gravitational theory.

(β) Inequalities in the Earth's time of rotation.

(γ) Action on the Moon of other forces than the gravitation of known bodies.

The second of these causes seems to be not at all improbable, in view of recent discoveries respecting variations of latitude, which can be accounted for only by minute changes in the Earth's axis of rotation. But the independent tests of this cause offered by the observed transits of Mercury do not show that it suffices to explain the observed inequalities in the motion of the Moon. Two other tests remain to be applied: the motion of the first satellite of Jupiter and observations of Mercury, Venus, and Mars. So far as evidence on these points has been brought out, the conclusion indicated is that inequalities of the Earth's rotation are real, but not large enough to produce the observed effects on the Moon's mean longitude. The subject is therefore still involved in doubt, and the first step toward the removal of this doubt is such a determination of the inequalities in the mean motion produced by the action of the planets, as shall place the subject on an undoubted basis.

CHAPTER I.

GENERAL THEORY.

§ I.

FUNDAMENTAL DIFFERENTIAL EQUATIONS FOR THE DISTURBED AND UNDISTURBED MOTION OF THREE BODIES.

For convenient reference, a part of the notation used throughout this paper is given here. We put

$m_1, m_2, m_3, m_4,$ the masses of the Sun, Earth, Moon, and planet, respectively;

$x, y, z,$ the rectangular coordinates of the Moon relative to the Earth;

$x', y', z',$ the rectangular coordinates of the Sun relative to the center of gravity of the Earth and Moon;

$$\mu = \frac{m_3}{m_2+m_3}$$

$$\mu' = \frac{m_2+m_3}{m_1+m_2+m_3}$$

$$\mu_1 = m_1\,\mu'$$

$$\mu_2 = m_2\,\mu$$

$X_4, Y_4, Z_4,$ etc., the rectangular coordinates of the planet relatively to the center of gravity of the Sun, Earth, and Moon;

$x_4, y_4, z_4, r_4,$ the rectangular coordinates and radius vector of the planet referred to the Sun;

$r,$ the radius vector of the Moon, so that

$$r^2 = x^2+y^2+z^2;$$

$r',$ the Sun's radius vector from center of gravity of Earth and Moon;

$$r'^2 = x'^2+y'^2+z'^2;$$

$\rho,$ the distance of the planet from the center of gravity of the Earth and Moon;

$a, e, \pi,$ etc., the elements of the Moon's orbit round the Earth, the major axes being determined by the condition $a^3 n^2 =$ sum of masses;

a', e', π', etc., the elements of the relative orbit of the Sun round the center of gravity of the Earth and Moon;

n, n', the mean motions of the Moon and Sun, respectively, as actually observed;

Ω, the potential, or sum of the products of each pair of masses divided by the distance, when only three bodies are considered;

R, the terms of the potential added by the planet;

$P = \Omega + R$, the entire potential;

x_1, y_1, z_1, x'_1, etc., the derivatives of x, y, z, x', etc., as to the time.

If we suppose, for the present, ξ_i, η_i, and ζ_i (i having in succession the values 1, 2, 3, 4), to represent the rectangular coordinates of the four bodies referred to any arbitrary fixed origin, and ξ_0, η_0, and ζ_0 to represent the coordinates of the center of gravity of the Sun, Earth and Moon, the differential equations of the motion of the three bodies will be

$$m_i \frac{d^2 \xi_i}{dt^2} = \frac{dP}{d\xi_i};\quad m_i \frac{d^2 \eta_i}{dt^2} = \frac{dP}{d\eta_i}, \text{ etc.} \tag{1}$$

$$(i = 1, 2, 3)$$

Both sides of these equations admit of being completely expressed in terms of the three relative coordinates already defined. The expressions for the relative coördinates x, y, z, x', y', z' in terms of the absolute coordinates are

$$\begin{aligned}
x &= \xi_3 - \xi_2 \\
y &= \eta_3 - \eta_2 \\
z &= \zeta_3 - \zeta_2 \\
x' &= \xi_1 - (1-\mu)\xi_2 - \mu\,\xi_3 \\
y' &= \eta_1 - (1-\mu)\eta_2 - \mu\,\eta_3 \\
z' &= \zeta_1 - (1-\mu)\zeta_2 - \mu\zeta_3
\end{aligned} \tag{2}$$

Differentiating these equations twice, and substituting the values of the second derivatives in (1) we find

$$\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{1}{m_3}\frac{dP}{d\xi_3} - \frac{1}{m_2}\frac{dP}{d\xi_2} \\
\frac{d^2x'}{dt^2} &= \frac{1}{m_1}\frac{dP}{d\xi_1} - \frac{1}{m_2+m_3}\left(\frac{dP}{d\xi_2} + \frac{dP}{d\xi_3}\right)
\end{aligned} \tag{3}$$

with similar equations in y, y', z and z'.

The four absolute coordinates $\xi_1, \xi_2, \xi_3, \xi_4$ admit of being completely expressed as a function of the four quantities x, x', ξ_0, and X_4. If, therefore, we suppose P expressed in terms of the last four quantities, we may put, in (3)

$$\frac{dP}{d\xi_i}=\frac{dP}{dx}\frac{dx}{d\xi_i}+\frac{dP}{dx'}\frac{dx'}{\partial\xi_i}+\frac{dP}{dX_4}\frac{dX_4}{d\xi_i}+\frac{dP}{d\xi_0}\frac{d\xi_0}{d\xi_i} \qquad (4)$$

the last term, however, being identically zero, because ξ_0 disappears from P.

The complete expressions for the coordinates ξ are

$$\begin{aligned}\xi_1&=\xi_0+\mu' x'\\ \xi_2&=\xi_0-(1-\mu')x'-\mu x\\ \xi_3&=\xi_0-(1-\mu')x'+(1-\mu)x\\ \xi_4&=\xi_0+X_4\end{aligned} \qquad (5)$$

Representing the distance of the bodies numbered i and j by $\rho_{i,j}$, the complete expression for P is

$$P=\Sigma^2_{i,j}\frac{m_i m_j}{\rho_{ij}}$$

the general value of $\rho_{i,j}$ being derived from

$$\rho^2_{i,j}=(\xi_i-\xi_j)^2+(\eta_i-\eta_j)^2+(\zeta_i-\zeta_j)^2$$

The expressions for the six values of $\rho_{i,j}$ in terms of the relative coordinates are from (5)

$$\begin{aligned}\rho^2_{12}&=(x'+\mu x)^2+(y'+\mu y)^2+(z'+\mu z)^2\\ \rho^2_{13}&=(x'-(1-\mu)x)^2+(y'-(1-\mu)y)^2+(z'-(1-\mu)z)^2\\ \rho^2_{14}&=(X_4-\mu' x')^2+(Y_4-\mu' y')^2+(Z_4-\mu' z')^2\\ \rho^2_{23}&=r^2=x^2+y^2+z^2\\ \rho^2_{24}&=(X_4+(1-\mu')x'+\mu x)^2+(Y_4+(1-\mu')y'+\mu y)^2\\ &\quad+(Z_4+(1-\mu')z'+\mu z)^2\\ \rho^2_{34}&=(X_4+(1-\mu')x'-(1-\mu)x)^2+(Y_4+(1-\mu')y'-(1-\mu)y)^2\\ &\quad+(Z^4+(1-\mu')Z'-(1-\mu)z)^2\end{aligned} \qquad (6)$$

From (2), combined with the value of X_4 derived from the expression (5), namely,

$$X_4=\xi_4-\frac{m_1\xi_1+m_2\xi_2+m_3\xi_3}{m_1+m_2+m_3}$$

we find the partial derivatives

$$\frac{dx}{d\xi_1}=0 \qquad \frac{dx}{d\xi_2}=-1 \qquad \frac{dx}{d\xi_3}=1$$

$$\frac{dx'}{d\xi_1}=1 \qquad \frac{dx'}{d\xi_2}=\mu-1 \qquad \frac{dx'}{d\xi_3}=-\mu$$

$$\frac{dX_4}{d\xi_1}=-\frac{m_1}{m_1+m_2+m_3} \qquad \frac{dX_4}{d\xi_2}=-\frac{m_2}{m_1+m_2+m_3} \qquad \frac{dX_4}{d\xi_3}=-\frac{m_3}{m_1+m_2+m_3}$$

These values being substituted in the general expression (4), we find for the values of the derivatives in (3)

$$\frac{d\,\mathrm{P}}{d\,\xi_1}=\frac{d\,\mathrm{P}}{d\,x'}-\frac{m_1}{m_1+m_2+m_3}\frac{d\,\mathrm{P}}{d\,\mathrm{X}_4}$$

$$\frac{d\,\mathrm{P}}{d\,\xi_2}=-\frac{d\,\mathrm{P}}{d\,x}+(\mu-1)\frac{d\,\mathrm{P}}{d\,x'}-\frac{m_2}{m_1+m_2+m_3}\frac{d\,\mathrm{P}}{d\,\mathrm{X}_4}$$

$$\frac{d\,\mathrm{P}}{d\,\xi_3}=\frac{d\,\mathrm{P}}{d\,x}-\mu\frac{d\,\mathrm{P}}{d\,x'}-\frac{m_3}{m_1+m_2+m_3}\frac{d\,\mathrm{P}}{d\,\mathrm{X}_4}$$

By the substitution of these values and the corresponding ones for the other two coordinates equations (3) become

$$\begin{aligned}
\mu_2\frac{d^2x}{d\,t^2}&=\frac{d\,\mathrm{P}}{d\,x}\\
\mu_2\frac{d^2y}{d\,t_2}&=\frac{d\,\mathrm{P}}{d\,y}\\
\mu_2\frac{d^2z}{d\,t^2}&=\frac{d\,\mathrm{P}}{d\,z}\\
\mu_1\frac{d^2x'}{d\,t^2}&=\frac{d\,\mathrm{P}}{d\,x'}\\
\mu_1\frac{d^2y'}{d\,t^2}&=\frac{d\,\mathrm{P}}{d\,y'}\\
\mu_1\frac{d^2z'}{d\,t^2}&=\frac{d\,\mathrm{P}}{d\,z'}
\end{aligned}\tag{7}$$

where P is to be conceived as expressed in the form

$$\mathrm{P}=f\,(x,y,z,x',y',z',\mathrm{X}_4,\mathrm{Y}_4,\mathrm{Z}_4)$$

by the substitution (5).

These are the fundamental differential equations of the problem. As thus presented, the coordinates of the planet X_4, Y_4, and Z_4, are supposed to be given in terms of the time. The differential equations for these coordinates, supposing the planet to be acted upon only by the Sun, Earth, and Moon, would be of the same form, and if joined to (7), the whole would represent the differential equations of the problem of four bodies. In reality, however, we should add nothing to the astronomical completeness of our problem by considering it as a problem of four bodies, because, if we regard the coordinates of the planet as depending on the attraction of the Earth and Moon, we must equally regard them as depending on the action of the other planets. We shall therefore regard our problem as that of the disturbed motion of three bodies, the position of the disturbing body being supposed to be given independently of the integrations by which the relative coordinates of the three bodies are determined.

§ 2.

REDUCED FORM OF THE POTENTIAL.

Let us first consider the principal portion Ω of the potential P, namely,

$$\Omega=\frac{m_1 m_2}{\rho_{1,2}}+\frac{m_1 m_3}{\rho_{1,3}}+\frac{m_2 m_3}{\rho_{2,3}}$$

the three denominators having the values given in equations (6). The coordinates x, y, and z being very small in comparison with x', y', and z', the first two terms of Ω may be developed in powers of their ratios. The first approximation will be the value which these terms assume when we suppose x, y, and z to vanish. Putting

$$r'^2=x'^2+y'^2+z'^2$$
$$r^2=x^2+y^2+z^2=\rho^2_{2,3},$$

we find for a portion of Ω, which we shall call Ω_0,

$$\Omega_0=\frac{m_1(m_2+m_3)}{r'}+\frac{m_2 m_3}{r} \tag{8}$$

while the remaining terms of Ω will be

$$\Omega_1=m_1\left\{m_2\left(\frac{1}{\rho_{1,2}}-\frac{1}{r'}\right)+m_3\left(\frac{1}{\rho_{1,3}}-\frac{1}{r'}\right)\right\}$$

Now let us put

W, the angular distance of the Sun and Moon, as seen from the center of gravity of the Earth and Moon. We shall then have from (6)

$$\rho^2_{1,2}=r'^2\left\{1+2\,\mu\,\frac{r}{r'}\cos \mathrm{W}+\mu^2\frac{r^2}{r'^2}\right\}$$

$$\rho^2_{1,3}=r'^2\left\{1-2\,(1-\mu)\,\frac{r}{r'}\cos \mathrm{W}+(1-\mu)^2\frac{r^2}{r'^2}\right\}$$

Developing the power of $-\frac{1}{2}$ of these expressions to terms of the second order in $\frac{r}{r'}$ we find

$$\frac{1}{\rho_{1,2}}=\frac{1}{r'}\left\{1-\mu\frac{r}{r'}\cos \mathrm{W}+\frac{1}{2}\mu^2\frac{r^2}{r'^2}(3\cos^2 \mathrm{W}-1)\right\}$$

$$\frac{1}{\rho_{1,3}}=\frac{1}{r'}\left\{1+(1-\mu)\frac{r}{r'}\cos \mathrm{W}+\frac{1}{2}(1-\mu)^2\frac{r^2}{r'^2}(3\cos^2 \mathrm{W}-1)\right\}$$

Substituting these values in Ω_1, and noticing that by the expression for μ, namely,

$$\mu=\frac{m_3}{m_2+m_3}$$

we have

$$m_2 \mu - m_3 (1 - \mu) = 0$$

$$m_2 \mu^2 + m_3 (1 - \mu)^2 = \frac{m_2 m_3}{m_2 + m_3} = \mu_2$$

we find

$$\Omega_1 = \frac{1}{2} \frac{m_1 m_2 m_3}{m_2 + m_3} \frac{r^2}{r'^3} (3 \cos^2 W - 1) \tag{9}$$

The expressions (8) and (9) make up the complete value of Ω when terms of the third and higher orders with respect to $\frac{r}{r'}$ are omitted.

The remaining terms of P are those depending on the mass of the disturbing planet, the expression being

$$R = m_4 \left(\frac{m_1}{\rho_{1,4}} + \frac{m_2}{\rho_{2,4}} + \frac{m_3}{\rho_{3,4}} \right)$$

For convenience in subsequent developments we shall substitute for the first term the integral of its derivatives with respect to the variables x', y', z', which we do because only these derivatives enter into the fundamental differential equations, and $\rho_{1,4}$ does not contain any of the quantities x, y, or z. This term of R may then be replaced by the expression

$$m_1 m_4 \int \left(\frac{d \frac{1}{\rho_{1,4}}}{d x'} d x' + \frac{d \frac{1}{\rho_{1,4}}}{d y'} d y' + \frac{d \frac{1}{\rho_{1,4}}}{d z'} d z' \right)$$

Substituting the value of $\rho_{1,4}$ in (6), and putting after differentiation

$$x_4, y_4, z_4, r_4$$

for the coordinates of the planet relative to the Sun, or

$$x_4 = X_4 - \mu' x', \text{ etc.},$$

this integral becomes

$$\int \frac{m_1 m_4 \mu'}{\rho^3_{1,4}} (x_4 d x' + y_4 d y' + z_4 d z')$$

and may be expressed in the form

$$m_1 m_4 \mu' \frac{x_4 x' + y_4 y' + z_4 z'}{r^3_4} \tag{10}$$

provided that in forming the partial derivatives we regard the coordinates of the planet relative to the Sun as independent of x', y', and z'.

The remaining two terms of R may be developed in the same way that we have just developed Ω_1. If we put

ρ, the distance of the planet from the center of gravity of the Earth and Moon, and for brevity,

ξ, η, ζ, the coordinates of the planet relative to the same origin, we have

$$\rho^2_{2,4} = \rho^2 + 2\,\mu\,(\xi\,x + \eta\,y + \zeta\,z) + \mu^2\,r^2$$
$$\rho^2_{3,4} = \rho^2 - 2\,(1-\mu)\,(\xi\,x + \eta\,y + \zeta\,z) + (1-\mu)^2\,r^2$$

Taking the power $-\frac{1}{2}$ and putting, for brevity,

$$\varDelta = \xi\,x + \eta\,y + \zeta\,z$$

we find for the principal terms

$$\frac{1}{\rho_{2,4}} = \frac{1}{\rho} - \frac{1}{\rho^3}\left(\mu\,\varDelta + \frac{1}{2}\,\mu^2\,r^2\right) + \frac{3\,\mu^2\,\varDelta^2}{2\,\rho^5} + \ \ldots$$
$$\frac{1}{\rho_{3,4}} = \frac{1}{\rho} - \frac{1}{\rho^3}\left((\mu-1)\,\varDelta + \frac{1}{2}\,(1-\mu)^2\,r^2\right) + \frac{3\,(1-\mu)^2\,\varDelta^2}{2\,\rho^5} + \ \ldots$$

Multiplying the first of these by $m_4\,m_2$, and the second by $m_4\,m_3$, and adding, we find for the corresponding terms of R

$$m_4\left\{\frac{m_2+m_3}{\rho} - \frac{\mu_2\,r^2}{2\,\rho^3} + \frac{3\,\mu_2\,\varDelta^2}{2\,\rho^5} + \ \ldots\ \right\} \tag{11}$$

The first term is subject to the inconvenience that ρ is there to be regarded in differentiating with respect to x' as a function of the coordinates of the Earth and planet relative to the center of gravity of the Sun, Earth, and Moon, and therefore as a function of x'. To avoid this inconvenience, and introduce the coordinates of the planet relative to the Sun, we shall make the same transformation we made on the first term of R, putting instead of $\frac{1}{\rho}$ the expression

$$\int\left(\frac{d\frac{1}{\rho}}{dx'}\,d\,x' + \frac{d\frac{1}{\rho}}{d\,y'}\,d\,y' + \frac{d\frac{1}{\rho}}{d\,z'}\,d\,z'\right) = -\int\frac{1-\mu'}{\rho^3}\,(\,\xi\,d\,x' + \eta\,d\,y' + \zeta\,d\,z'\,)$$

When we introduce instead of X_4, Y_4, Z_4 the coordinates of the planet relatively to the Sun we have

$$X_4 = x_4 - (1-\mu')\,x',\ \text{etc.},$$

and hence

$$\xi = x_4 + x',\ \ \eta = y_4 + y',\ \ \zeta = z_4 + z'\ .$$

Substituting these values in the integral it becomes

$$-\int\frac{1-\mu'}{\rho^3}\,\rho\,d\,\rho = \frac{1-\mu'}{\ } \tag{12}$$

so that the first term of (11) is simply to be multiplied by $1-\mu'$, and ρ to be regarded as a function of the heliocentric position of the planet only.

Collecting the various terms of R in (10) and (11) as modified by (12) we have its complete transformed expression. The fundamental differential equations (7) now become

$$
\left.\begin{aligned}
\mu_1\frac{d^2x'}{dt^2}&=\frac{d\Omega}{dx'}+\frac{dR}{dx'}\\
\mu_1\frac{d^2y'}{dt^2}&=\frac{d\Omega}{dy'}+\frac{dR}{dy'}\\
\mu_1\frac{d^2z'}{dt^2}&=\frac{d\Omega}{dz'}+\frac{dR}{dz'}\\
\mu_2\frac{d^2x}{dt^2}&=\frac{d\Omega}{dx}+\frac{dR}{dx}\\
\mu_2\frac{d^2y}{dt^2}&=\frac{d\Omega}{dy}+\frac{dR}{dy}\\
\mu_2\frac{d^2z}{dt^2}&=\frac{d\Omega}{dz}+\frac{dR}{dz}
\end{aligned}\right\} \tag{13}
$$

Where

$$
\left.\begin{aligned}
\Omega &=\Omega_0+\Omega_1\\
\Omega_0&=\frac{m_1(m_2+m_3)}{r'}+\frac{m_2m_3}{r}\\
\Omega_1&=\frac{1}{2}m_1\mu_2\frac{r^2}{r'^3}\left(3\cos^2W-1\right)\\
R &=m_4\mu_1\left(\frac{1}{\rho}+\frac{x_4x'+y_4y'+z_4z'}{r_4^3}\right)+\frac{1}{2}m_4\mu_2\left(\frac{3\Delta^2}{\rho^5}-\frac{r^2}{\rho^3}\right)
\end{aligned}\right\} \tag{14}
$$

in which we have put

$$\mu_1=\frac{m_1(m_2+m_3)}{m_1+m_2+m_3}=m_1\mu'$$

$$\mu_2=\frac{m_2m_3}{m_2+m_3}=m_2\mu$$

$$\Delta =x(x_4+x')+y(y_4+y')+z(z_4+z')$$

It will be remarked that the variable part of Ω_1 is identical with the force function for the action of the Sun in the lunar theory, and that R is closely analogous to the disturbing function for the action of a planet in DELAUNAY'S theory. It is also to be noted that Ω_1 and R are developed in powers of the ratio of the mean distance of the

Moon to that of the Sun or planet, and that only the first terms of the development are included in the expressions (14).

§ 3.

COMPARISON OF TWO METHODS OF INTEGRATING THE PRECEDING EQUATIONS.

The differential equations (13) of the preceding chapter, or some transformation of them, form the necessary starting point of all methods of treating the theory of the Moon's motion. It may also be held that the most rigorous method hitherto developed of treating the problem analytically, and now well known, is that resting upon the results of DELAUNAY'S system of integrating the equations in question, or upon some process equivalent to it.

The problem as actually considered by DELAUNAY was that of three bodies only, considered as material points. Even with this restriction he did not fully complete the solution, because he considered the relative elliptic orbit of the Sun to be performed around the center of the Earth, instead of the center of gravity of the Earth and Moon. So far as the Moon's motion is concerned, the additions required to reduce his solution to the actual case are extremely simple, so that for all practical purposes we may consider DELAUNAY'S theory to be complete as regards the Moon. The deviations of the path described by the center of gravity of the Earth and Moon from an elliptic orbit around the Sun are, from a theoretical point of view, necessary to complete the solution, but are too minute to be of importance in observational astronomy.

To complete the solution of the problem of the Moon's motion subsidiary actions have to be taken into account, such as those arising from the ellipticity of the Earth and from the action of the planets. It is to the two methods of effecting these subsidiary solutions that the attention of the reader is now called. One of these was doubtless that intended by DELAUNAY himself, and consists essentially in continuing the method adopted by DELAUNAY, considering the actions in question simply as perturbations additional to those produced by the Sun. This method has been followed by G. W. HILL on DELAUNAY'S lines in Vol. III of these papers. The methods of other investigators, as RADAU and NEVILL, may also be considered as founded substantially upon the same idea. This I shall designate hereafter as the first method.

In the second method, which it is the object of the present paper to develop, all the constants of integration of the problem of three bodies are considered as simultaneously variable under the action of a fourth body, a planet. In the actual application of this second method there is so close an analogy to the first that the two must be mutually helpful to each other. Their relations can be clearly shown by a comparative statement of the general plan followed in them.

In both methods the solution of the last three equations (13) is obtained by three successive steps. In the first step, which is common to both methods, we have, instead of these equations, those which result from putting Ω_0 for $\Omega + \mathrm{R}$, the value of the force function then used being

$$\Omega_0 = \frac{m_1(m_2+m_3)}{r'} + \frac{m_2 m_3}{r}$$

The results of this step are the theories of elliptic motion; the moon revolves around the Earth in an ellipse, as does the center of gravity of the Earth and Moon around the Sun. Thus in the solution the Moon's coordinates, x, y, and z, are represented as functions of six arbitrary constants, say

$$a_1, a_2, \quad . \quad . \quad . \quad a_6$$

and of the time; while the Sun's coordinates, x', y', and z', appear as functions of six other arbitrary constants, say

$$a'_1, a'_2, \quad . \quad . \quad . \quad a'_6$$

and of the time.

In this step the effect of the solution is that the twelve arbitrary constants of motion are completely separated into two classes, six appearing only in the expressions for the Moon's coordinates, and the other six only in those of the Sun.

In the second step $\Omega_0+\Omega_1$ is substituted for Ω_0 in the second members of the differential equations which determine the Moon's motion. Ω_1 is here a function of the coordinates of both bodies, and the value of its principal term is given in (14). The complete solution of this step is effected by regarding the twelve elements as variable. Since at the first step the Moon's coordinates contain only unaccented elements, which we may call lunar elements, and the Sun only the accented elements, which we may call solar elements, it follows that the LA GRANGIAN coefficients which pertain to the Moon will contain only the six lunar elements, and those which pertain to the Sun only the solar ones. The solution might then be made by successive approximations, the lunar elements being first regarded as constant in integrating the equations for the Sun's motion, and the solar elements as constant in integrating the equations for the Moon's motion. The values of the twelve elements obtained in this first approximation may then be substituted in the second members of all the equations, a second approximation made, and so on until a solution of the requisite accuracy is obtained.

In DELAUNAY'S theory, as in all others hitherto published, the attention being confined to the Moon's coordinates, the solution is effected only for the lunar elements, and moreover the solar elements are regarded as constant, it being practically unnecessary to consider the variations of the solar elements due to Ω_1. Successive approximations are, however, made in the case of the lunar elements. These elements, and hence the lunar coordinates, thus become functions of twelve arbitrary constants, six of which pertain to the Sun alone, and are not constants of integration, but are considered as entering into the differential equations of the Moon's motion. The other six are the constants of integration, and may be called lunar constants. We may represent these six lunar constants by the symbols

$$\alpha_1, \alpha_2 \quad . \quad . \quad . \quad . \quad \alpha_6$$

So far we have only the solution of the problem of three bodies, which, in the second method, is considered to be solved in advance. The divergence of the two methods now commences in the continuation of the solution. As a matter of fact, the solar elements, say

$$\alpha'_1, \alpha'_2 \quad . \quad . \quad . \quad . \quad \alpha'_6$$

become variable through the action of the planet. The determination of the variations offers no difficulty, their value being the same whichever method be adopted. But in the first method, when the coordinates of the Moon are considered, these solar elements are not regarded as constants of integration, but as quantities given *a priori*, and appearing in the differential equations. The LA GRANGIAN coefficients

$$[\alpha_i, \alpha_j] = \frac{d\,x}{d\,\alpha_i}\frac{d\,x_1}{d\,\alpha_j} - \frac{d\,x_1}{d\,\alpha_i}\frac{d\,x}{d\,\alpha_j} + \frac{d\,y}{d\,\alpha_i}\frac{d\,y_1}{d\,\alpha_j} - \text{etc.}$$

are therefore formed only for the combinations of the six lunar elements, and the perturbative function R appears in two parts, the one being the value of R, as found in (14), the other the value of $\delta\,\Omega_1$ arising from the perturbations of the solar elements through the action of the planet, which we may conceive expressed in the form

$$\delta\,\Omega_1 = \frac{d\,\Omega_1}{d\,\alpha'_1}\delta\,\alpha'_1 + \frac{d\,\Omega_1}{d\,\alpha'_2}\delta\,\alpha'_2 + \quad . \quad . \quad .$$

Thus in the final solution obtained in this way the Moon's coordinates will not appear as functions of the actual variable elements of the Sun, but of mean elements at some epoch, the entire effect of the perturbations produced by the planet being thrown upon the lunar elements.

In the second method, all twelve elements are regarded as simultaneously variable, and the LA GRANGIAN coefficients are formed accordingly, so that the lunar coordinates are in the first plane considered as functions of all twelve varying elements.

The final forms of the solution by the two methods may now be seen and compared.

In the results of DELAUNAY's work, where only three bodies are considered, the Moon's coordinates are expressed as a function of the twelve constants

$$\alpha_1, \alpha_2 \quad . \quad . \quad . \quad . \quad \alpha_6, \alpha'_1, \alpha'_2 \quad . \quad . \quad . \quad . \quad \alpha'_6$$

When the action of the planet is included by HILL's continuation of DELAUNAY's process, the Moon's coordinates are regarded as still the same functions of these twelve elements, the accented ones remaining constant, and the others becoming variable.

In the new method the coordinates are also regarded as still the same functions of these elements, but all twelve of the elements are regarded as variable.

We may now construct the equations of condition, which must be satisfied in order that the two methods may give the same result. Let us represent by the symbol δ' the perturbations of the elements by the first method, and by δ those by the second. The corresponding perturbations of any coordinate v will then be

$$\text{I.}\quad \delta v = \frac{dv}{d\alpha_1}\delta'\alpha_1 + \frac{dv}{d\alpha_2}\delta'\alpha_2 + \quad . \quad . \quad . \quad + \frac{dv}{d\alpha_6}\delta'\alpha_6$$

$$\text{II.}\quad \delta v = \frac{dv}{d\alpha_1}\delta\alpha_1 + \quad . \quad . \quad . \quad + \frac{dv}{d\alpha_6}\delta\alpha_6 + \frac{dv}{d\alpha'_1}\delta\alpha'_1 + \quad . \quad . \quad . \quad + \frac{dv}{d\alpha'_6}\delta\alpha'_6$$

The equality of these perturbations gives the relation

$$\Sigma\frac{dv}{d\alpha_i}\left(\delta'\alpha_i - \delta\alpha_i\right) = \Sigma\frac{dv}{d\alpha'_i}\delta\alpha'_i \qquad (15)$$

$$(i = 1, 2, \quad . \quad . \quad . \quad . \quad 6)$$

which is the required relation.

§ 4.

FUNDAMENTAL EQUATIONS OF THE METHOD.

The present paper being devoted wholly to the development of the second of the preceding methods, we have to consider further the general method of proceeding. We recall that in the general La Grangian method, which alone will be used, it is presupposed that a solution is obtained giving the values of the coordinates in terms of the requisite number of arbitrary quantities, such as shall completely satisfy the differential equations when certain terms of the potential are omitted. A solution of the case when the omitted terms are taken account of is then obtained by a system of ordinary differential equations, in which the variables are the arbitrary elements introduced by the preceding solution. There is no limit to the number of times by which this process may be successively repeated. The effect of each new term of the potential introduced is taken account of by the integration of a system of differential equations in which the arbitrary constants of the preceding solution appear as the variables. Delaunay's classic work forms a beautiful example of the operations by which the equations of the Moon's motion, as affected by the action of the Sun, may thus be integrated.

The problem with which the present paper is ultimately concerned is not that of the motion of the Moon as affected by the action of the Sun, but as affected by the action of the planets. This action is involved only in the term R of the potential. Hence the sole object of the present paper is to to determine the variations of the lunar, and incidentally of the solar elements produced by the term in question, the equations being completely integrated for the case when R is omitted. The general nature of the process is so well known that no development is necessary. We shall therefore merely present a résumé of the necessary equations.

Firstly. We presuppose that the values of the six coordinates x, y, z, x', y', z', are expressed in terms of the time and of twelve arbitrary constants, which we may represent in the most general way by the symbols

$$\alpha_1, \alpha_2, \quad . \quad . \quad . \quad . \quad \alpha_{12},$$

in such way as to completely satisfy the six equations

$$\begin{array}{ll} \mu_2 \frac{d^2 x}{d t^2} = \frac{d \Omega}{d x} & \mu_1 \frac{d^2 x'}{d t^2} = \frac{d \Omega}{d x'} \\[2ex] \mu_2 \frac{d^2 y}{d t^2} = \frac{d \Omega}{d y} & \mu_1 \frac{d^2 y'}{d t^2} = \frac{d \Omega}{d y'} \\[2ex] \mu_2 \frac{d^2 z}{d t^2} = \frac{d \Omega}{d z} & \mu_1 \frac{d^2 z'}{d t^2} = \frac{d \Omega}{d z'} \end{array} \tag{16}$$

Let us represent these expressions in the form

$$\begin{array}{ll} x = f_1 & x' = \mathrm{F}_1 \\ y = f_2 & y' = \mathrm{F}_2 \\ z = f_3 & z' = \mathrm{F}_3 \end{array} \tag{17}$$

where f and F are functions of the twelve arbitrary constants and of t.

Secondly. We construct the conjugate functions x_1, y_1, z_1, x'_1, y'_1, z'_1 by the equations

$$\begin{array}{ll} x_1 = \mu_2 \frac{d f_1}{d t} & x'_1 = \mu_1 \frac{d \mathrm{F}_1}{d t} \\[2ex] y_1 = \mu_2 \frac{d f_2}{d t} & y'_1 = \mu_1 \frac{d \mathrm{F}_2}{d t} \\[2ex] z_1 = \mu_2 \frac{d f_3}{d t} & z'_1 = \mu_1 \frac{d \mathrm{F}_3}{d t} \end{array} \tag{18}$$

The variations of the twelve elements when R is taken into account must then satisfy a simultaneous system of twelve differential equations, of which four are as follows:

$$\begin{array}{l} \frac{d x}{d \alpha_1} \frac{d \alpha_1}{d t} + \frac{d x}{d \alpha_2} \frac{d \alpha_2}{d t} + \;.\;.\;.\;.\; + \frac{d x_{12}}{d \alpha_{12}} \frac{d \alpha_{12}}{d t} = 0 \\[2ex] \frac{d x_1}{d \alpha_1} \frac{d \alpha_1}{d t} + \frac{d x_1}{d \alpha_2} \frac{d \alpha_2}{d t} + \;.\;.\;.\;.\; + \frac{d x_1}{d \alpha_{12}} \frac{d \alpha_{12}}{d t} = \frac{d \mathrm{R}}{d x} \\[2ex] \frac{d x'}{d \alpha_1} \frac{d \alpha_1}{d t} + \frac{d x'}{d \alpha_2} \frac{d \alpha_2}{d t} + \;.\;.\;.\;.\; + \frac{d x'}{d \alpha_{12}} \frac{d \alpha_{12}}{d t} = 0 \\[2ex] \frac{d x'_1}{d \alpha_1} \frac{d \alpha_1}{d t} + \frac{d x'_1}{d \alpha_2} \frac{d \alpha_2}{d t} + \;.\;.\;.\;.\; + \frac{d x'_1}{d x_{12}} \frac{d \alpha_{12}}{d t} = \frac{d \mathrm{R}}{d x'} \end{array} \tag{19}$$

while the remaining eight are formed by writing y and z in the place of x, and need not therefore be actually written.

From these equations are derived, by a well known linear combination, the La Grangian equations for the variation of elements in the following form. We put

$$[a_i, a_j] = \Sigma \left(\frac{d x}{d a_i} \frac{d x_1}{d \alpha_j} - \frac{d x}{d a_j} \frac{d x_1}{d a_i} \right) \tag{20}$$

the sign Σ indicating a summation of the terms which arise by writing x, y, z, x', y', z' in succession for x.

We also suppose that R becomes a function of the elements α_i, and of the time by the substitution of the values of the coordinates as functions of these quantities. The La Grangian equations for the variation of the elements then take the following form:

$$[\alpha_i\,\alpha_1]\frac{d\,\alpha_1}{d\,t}+[\alpha_i,\alpha_2]\frac{d\,\alpha_2}{d\,t}+\quad .\ .\ .\ .\ .\quad+[\alpha_i,\alpha_{12}]\frac{d\,\alpha_{12}}{d\,t}=\frac{d\,\mathrm{R}}{d\,\alpha_i}\qquad(21)$$

$$(i=1;\ 2;\ 3;\ .\ .\ .\ .\ .\ .\ 12)$$

In determining these La Grangian coefficients I make use of the known theorem that they are constants. This theorem rests, however, upon the supposition that the expressions for the coordinates actually satisfy the differential equations. As a matter of fact they are developed in series, so that it is impossible that this condition shall be rigorously fulfilled. It is therefore important to determine finally to what degree of approximation the constancy of the coefficients in question is assured. For our present purpose I have deemed it sufficient to develop the constant portions of the coefficients so far as the degree of approximation to which the solution is carried would admit.

This theorem of the constancy of the coefficients enables their values to be formed with much greater facility than if it were necessary to actually perform all the multiplications which enter into the equation (20).

In considering the general theory a yet farther simplification may be made. It is known that if we take six of the elements, which we may call

$$l_1,\ l_2,\ .\ .\ .\ .\ l_6$$

at pleasure, we can always substitute for the other six certain functions of them,

$$c_1,\ c_2\ .\ .\ .\ .\ c_6$$

such that the La Grangian coefficients shall reduce to zero in all the combinations except those six in which an l_i is coordinated with its corresponding c_i, in which case the combination will be reduced to unity. We shall make frequent use of this theorem in the course of the present investigation.

Such a selection of elements I shall hereafter call *canonical.*

§ 5.

THE PROBLEM OF THREE BODIES.

The preliminary step now is to express the six coordinates, x, y, z, x', y', z', in terms of twelve arbitrary constants, so as to satisfy the differential equations (16).

We recall that two processes of integration are necessary in the preliminary investigation, namely:

1. We integrate omitting Ω_1 in the equations (13), writing Ω_0 for $\Omega+\mathrm{R}$.
2. We include Ω_1, but omit R, as in the equations (16).

The first step involves no difficulty, since it gives rise only to the equations of elliptic motion—

(*a*) Of the Moon around the Earth;

(*b*) Of the center of gravity of the Earth and Moon around the Sun.

Putting in a general way

$$c_1,\ c_2,\ c_3,\ l_1,\ l_2,\ l_3$$

for a set of canonical elliptic elements pertaining to the Moon, and

$$c'_1,\ c'_2,\ c'_3,\ l'_1,\ l'_2,\ l'_3$$

for those pertaining to the Sun, we shall then have, as the result of putting $\Omega_1 = 0$

$$\begin{aligned} &x,\ y,\ \text{and}\ z = f(c_1,\ c_2,\ c_3,\ l_1,\ l_2,\ l_3,\ t)\ \text{and} \\ &x',\ y',\ \text{and}\ z' = f(c'_1,\ c'_2,\ c'_3,\ l'_1,\ l'_2,\ l'_3,\ t) \end{aligned} \tag{22}$$

Next, to take account of Ω_1, we substitute in Ω_1, for x, y, z, x', y', z', their expressions (22). Then, using canonical elements, the form (21) will become

$$\begin{aligned} \frac{d c_i}{d t} &= \frac{d \Omega_1}{d l_i} \qquad & \frac{d l_i}{d t} &= -\frac{d \Omega_1}{d c_i} \\ & & & \qquad (i = 1, 2, 3) \\ \frac{d c'_i}{d t} &= \frac{d \Omega_1}{d l'_i} \qquad & \frac{d l'_i}{d t} &= -\frac{d \Omega_1}{d c'_i} \end{aligned} \tag{23}$$

These equations are now to be integrated by successive approximations. We have first to integrate the equations for c and l on the supposition that c' and l' are constant. Next, by substituting these values of c and l in the equations for c' and l' and integrating, we obtain approximate expressions for these quantities. If necessary these new values of c' and l' can be employed to reintegrate the first set of equations, and so on.

The first integration has been performed by DELAUNAY in his great work *Théorie du mouvement de la Lune.* We may consider that, in this work, DELAUNAY has implicitly formed expressions c and l which substituted in (22) give values of x, y, and z that completely satisfy the equations

$$\mu_2 \frac{d^2 x}{d t^2} = \frac{d(\Omega_0 + \Omega_1)}{d x}\ \text{etc.} \tag{24}$$

on the supposition that c'_i and l'_i are treated as constants. More exactly, if we put, in (14)

$$\begin{aligned} \Omega'_0 &= \frac{m_2 m_3}{r} \\ \Omega''_0 &= \frac{m_1 (m_2 + m_3)}{r'} \end{aligned} \tag{25}$$

§ 6.

CONSIDERATIONS ON THE FORM AND ORDER OF MAGNITUDE OF THE EXPRESSIONS WHICH ENTER INTO THE PRECEDING THEORY.

The fifteen coefficients $[\alpha_i, \alpha_j]$ which enter into the equations (21) are each divisible into two parts, one of which is formed by differentiating the lunar coordinates x, y, and z, the other by differentiating the solar coordinates x', y', and z'. If we put

$$\begin{aligned}[\alpha_i\,\alpha_j]_1 = {} & \frac{dx}{d\alpha_i}\frac{dx_1}{d\alpha_j} + \frac{dy}{d\alpha_i}\frac{dy_1}{d\alpha_j} + \frac{dz}{d\alpha_i}\frac{dz_1}{d\alpha_j} \\ & - \frac{dx}{d\alpha_j}\frac{dx_1}{d\alpha_i} - \frac{dy}{d\alpha_j}\frac{dy_1}{d\alpha_i} - \frac{dz}{d\alpha_j}\frac{dz_1}{d\alpha_i}\end{aligned} \tag{30}$$

$$\begin{aligned}[\alpha_i\,\alpha_j]_2 = {} & \frac{dx'}{d\alpha_i}\frac{dx'_1}{d\alpha_j} + \frac{dy'}{d\alpha_i}\frac{dy'_1}{d\alpha_j} + \frac{dz'}{d\alpha_i}\frac{dz'_1}{d\alpha_j} \\ & - \frac{dx'}{d\alpha_j}\frac{dx'_1}{d\alpha_i} - \frac{dy'}{d\alpha_j}\frac{dy'_1}{d\alpha_i} - \frac{dz'}{d\alpha_j}\frac{dz'_1}{d\alpha_i}\end{aligned} \tag{31}$$

$$\left(\frac{dR}{d\alpha_i}\right)_1 = \frac{dR}{dx}\frac{dx}{d\alpha_i} + \frac{dR}{dy}\frac{dy}{d\alpha_i} + \frac{dR}{dz}\frac{dz}{d\alpha_i} \tag{32}$$

$$\left(\frac{dR}{d\alpha_i}\right)_2 = \frac{dR}{dx'}\frac{dx'}{d\alpha_i} + \frac{dR}{dy'}\frac{dy'}{d\alpha_i} + \frac{dR}{dz'}\frac{dz'}{d\alpha_i} \tag{33}$$

we may form by means of the expression (17) the two equations

$$\sum_{j=1}^{j=12} [\alpha_i, \alpha_j]_1 \frac{d\alpha_j}{dt} = \left(\frac{dR}{d\alpha_i}\right)_1 \tag{34}$$

$$\sum_{j=1}^{j=12} [\alpha_i, \alpha_j]_2 \frac{d\alpha_j}{dt} = \left(\frac{dR}{d\alpha_i}\right)_2 \tag{35}$$

and the sum of these two equations will make up the equation (21).

We may divide the twelve elements α into two classes, which may be designated *lunar* and *solar* elements. The first class comprise those which, when the Sun's disturbing force vanishes, merge into the elements of the Moon's elliptic motion around the Earth; the second, those which, when the mass of the Moon vanishes, merge into the elements of the Earth's elliptic motion around the Sun.

Let us next investigate the order of magnitude of these several coefficients and of $\left(\frac{dR}{d\alpha}\right)_1$ and $\left(\frac{dR}{d\alpha}\right)_2$. In the expressions (17) for the solar coordinates we are to distinguish two portions, one a purely elliptic part containing only the six elements of the elliptic motion of the Sun, the other part containing the terms added when we take into account the distance of the Moon as compared with that of the Sun. The first part arises from the integration of the differential equations for x', y', z' when only the part Ω_0 of the potential is considered, and is equal to the mean distance of the Sun, a', multiplied by purely numerical ratios.

In the expressions for x, y, and z, all the terms containing solar elements will be multiplied by $\frac{n'}{n}$.

We shall use the symbols S, S′, etc., to represent in the most general way the sum of a series of sines and cosines of angles, each multiplied into purely numerical ratios. These symbols will not twice represent the same quantity, and may, in special cases, reduce to purely numerical constants. If we put for brevity

$$\begin{aligned} m &= \frac{n'}{n} \\ k &= \mu \frac{a^2}{a'^2} \end{aligned} \tag{36}$$

the expressions of the form (17) and (18) for x, x_1, etc., and their derivatives may be put in the form

$$\begin{aligned} x &= a\ (\mathrm{S} + m\,\mathrm{S}') \\ x' &= a'\ (\mathrm{S}' + k\,\mathrm{S}) \\ x_1 &= \mu_2\, a\, n\, (\mathrm{S} + m\,\mathrm{S}') \\ x'_1 &= \mu_1\, a'\, n'\ (\mathrm{S}' + k\,\mathrm{S}) \\ &\text{etc.} \qquad \text{etc.} \end{aligned} \tag{37}$$

If, in forming the derivatives of these expressions as to the elements $\alpha_1 \ldots a_{12}$, we choose for the latter purely numerical quantities, which we may always do by replacing a line element by its logarithm, then the derivatives in question will be of the same form and the same order of magnitude with x, x', etc., themselves, or more exactly with the terms of those quantities which contain the elements. If, then, as before we distinguish the solar elements by accents, the order of magnitude of the largest terms of several derivatives will be as follow:

$$\begin{aligned} \frac{d\,x}{d\,\alpha} &= a\,\mathrm{S} \\ \frac{d\,x_1}{d\,\alpha} &= \mu_2\, a\, n\,\mathrm{S} \\ \frac{d\,x'}{d\,\alpha} &= k\, a'\,\mathrm{S} \\ \frac{d\,x'_1}{d\,\alpha} &= \mu_1\, k\, a'\, n'\,\mathrm{S} \end{aligned} \tag{38}$$

$$\left.\begin{aligned}
\frac{dx}{d\alpha'} &= m\,a\,S' \\
\frac{dx_1}{d\alpha'} &= \mu_2\,m\,a\,n\,S' \\
\frac{dx'}{d\alpha'} &= a'\,S' \\
\frac{dx'_1}{d\alpha'} &= \mu_1\,a'\,n'\,S'
\end{aligned}\right\} \qquad (38)$$

The various classes of coefficients $[a_i, a_j]$ formed by (30) and (31) will therefore be of the following orders of magnitude:

$$\left.\begin{aligned}
[\alpha, \alpha_i]_1 &= \mu_2\,a^2\,n\,S \\
[\alpha', \alpha_i]_1 &= \mu_2\,m\,a^2\,n\,S \\
[\alpha, \alpha'_i]_1 &= \mu_2\,m\,a^2\,n\,S \\
[\alpha', \alpha'_i]_1 &= \mu_2\,m^2\,a^2\,n\,S \\
[\alpha, \alpha_i]_2 &= \mu_1 k^2\,a'^2\,n'\,S \\
[\alpha, \alpha'_i]_2 &= \mu_1\,k\,a'^2\,n'\,S \\
[\alpha', \alpha_i]_2 &= \mu_1\,k\,a'^2\,n'\,S \\
[\alpha', \alpha'_i]_2 &= \mu_1\,a'^2\,n'\,S
\end{aligned}\right\} \qquad (39)$$

Let us next revert to the perturbative function R as found in (14). We see that it is composed of two parts of very different magnitude. Considering all linear heliocentric coordinates as of the same order of magnitude, a', and the coordinates of the Moon as of the order of a, the order of magnitude of the two terms of R will be

$$R = \frac{m_4\,\mu_1}{a'}\,S + \frac{m_4\,\mu_2\,a^2}{a'^3}\,S$$

or, if we use coordinates instead of mean distances, the orders of magnitude will be represented by

$$R = \frac{m_4\,\mu_1}{x'}\,S + \frac{m_4\,\mu_2\,x^2}{x'^3}\,S$$

where the first term does not contain the Moon's coordinates.

From this equation we shall have, using the sign $=$ only to indicate identity of form and order of magnitude,

$$\frac{dR}{dx} = \frac{m_4\,\mu_2\,x}{x'^3}\,S = \frac{m_4\,\mu_2\,a}{a'^3}\,S$$

$$\frac{dR}{dx'} = \frac{m_4\,\mu_1}{x'^2}\,S + \frac{m_4\,\mu_2\,x^2}{x'^4}\,S = \frac{m_4\,\mu_1}{a'^2}\,S + \frac{m_4\,\mu_2\,a^2}{a'^4}\,S$$

and thence, from (32) and (33), having reference to the orders of magnitude of $\frac{dx}{d\alpha}$, etc.,

$$\begin{aligned}
\left(\frac{d\mathrm{R}}{d\alpha}\right)_1 &= \frac{m_4\,\mu_2\,a^2}{a'^3}\mathrm{S} = \frac{m_4\,\mu_1\,k}{a'}\mathrm{S} \\
\left(\frac{d\mathrm{R}}{d\alpha}\right)_2 &= \frac{m_4\,\mu_1\,k}{a'}\mathrm{S} \\
\left(\frac{d\mathrm{R}}{d\alpha'}\right)_1 &= \frac{m_4\,\mu_2\,m\,a^2}{a'^3}\mathrm{S} = \frac{m_4\,\mu_1\,m\,k}{a'}\mathrm{S} \qquad (40) \\
\left(\frac{d\mathrm{R}}{d\alpha'}\right)_2 &= \frac{m_4\,\mu_1}{a'}\mathrm{S}
\end{aligned}$$

The approximate values of μ_1 and μ_2 are

$$\mu_1 = m_2$$

$$\mu_2 = m_3$$

by which we shall replace them in R. The relations between the mean motions and mean distances give, approximately,

$$a^2\,n = \frac{m_2}{a\,n}$$

$$a'^2\,n' = \frac{m_1}{a'\,n'}$$

Substituting these values in the expressions (39) for the coefficients $[\alpha_1, \alpha_i]$, and representing by S_1, S_2, etc., U_1, U_2, etc., numerical quantities like those we have hitherto represented by the general symbol S, we have

$$\begin{aligned}
[\alpha, \alpha_i]_1 &= \frac{m_2\,m_3}{a\,n}\mathrm{U}_i^{(4)} \\
[\alpha, \alpha'_i]_1 &= \frac{m_2\,m_3}{a\,n}m\,\mathrm{U}_i^{(3)} \qquad (i = 1, 2\ \ldots\ 6) \\
[\alpha', \alpha_i]_1 &= \frac{m_2\,m_3}{a\,n}m\,\mathrm{U}_i^{(2)} \\
[\alpha', \alpha'_i]_1 &= \frac{m_2\,m_3}{a\,n}m^2\,\mathrm{U}_i^{(1)} \\
[\alpha, \alpha_i]_2 &= \frac{m_1\,m_2}{a'\,n'}k^2\,\mathrm{S}_i^{(4)} \qquad (41) \\
[\alpha, \alpha'_i]_2 &= \frac{m_1\,m_2}{a'\,n'}k\,\mathrm{S}_i^{(3)} \\
[\alpha', \alpha_i]_2 &= \frac{m_1\,m_2}{a'\,n'}k\,\mathrm{S}_i^{(2)} \\
[\alpha', \alpha'_i]_2 &= \frac{m_1\,m_2}{a'\,n'}\mathrm{S}_i^{(1)}
\end{aligned}$$

The expressions thus formed are to be substituted in the pair of general equations (34) and (35) which, it must be remembered, may belong to either of two classes, according as α_j there represents a solar (accented) element, or a lunar (unaccented) one. Let us consider the former case first. The right-hand members of these equations, taken from (40), may then be put in the form

$$\left(\frac{d\,\mathrm{R}}{d\,\alpha'}\right)_1 = m\,\frac{m_2\,m_4}{a'}\,k\,\mathrm{U}^{(5)} = \frac{m_3\,m_4}{a}\,m\,\frac{a^3}{a'^3}\,\mathrm{U}^{(5)}$$

$$\left(\frac{d\,\mathrm{R}}{d\,\alpha'}\right)_2 = \frac{m_2\,m_4}{a'}\left(\mathrm{S}^{(5)} + k\,\mathrm{S}^{(6)}\right)$$

The pair of equations then become

$$\frac{m_1\,m_2}{a'\,n'}\left\{\mathrm{S}_1^{(1)}\frac{d\,\alpha'_1}{d\,t} + \mathrm{S}_2^{(1)}\frac{d\,\alpha'_2}{d\,t} + \;.\;\;.\; + \mathrm{S}_6^{(1)}\frac{d\,\alpha'_6}{d\,t} + k\left(\mathrm{S}_1^{(2)}\frac{d\,\alpha_1}{d\,t} + \;.\;\;.\; + \mathrm{S}_6^{(2)}\frac{d\,\alpha_6}{d\,t}\right)\right\}$$

$$= \frac{m_2\,m_4}{a'}\left(\mathrm{S}^{(5)} + k\,\mathrm{S}^{(6)}\right)$$

$$\frac{m_2\,m_3}{a\,n}\left\{m^2\left(\mathrm{U}_1^{(1)}\frac{d\,\alpha'_1}{d\,t} + \;.\;\;.\; + \mathrm{U}_6^{(1)}\frac{d\,\alpha'_6}{d\,t}\right) + m\left(\mathrm{U}_1^{(2)}\frac{d\,\alpha_1}{d\,t} + \;.\;\;.\; + \mathrm{U}_1^{(2)}\frac{d\,\alpha_6}{d\,t}\right)\right\} \tag{42}$$

$$= \frac{m_3\,m_4}{a}\,m\,\frac{a^3}{a'^3}\,\mathrm{U}^{(5)}$$

The sum of these two equations gives one of the rigorous **La Grangian** equations for the variations of the elements, in which each coefficient of the differential of an element is a constant. The coefficient of $\frac{d\,\alpha'_i}{d\,t}$ in the sum is

$$\frac{m_1\,m_2}{a'\,n'}\,\mathrm{S}_i^{(1)} + \frac{m_2\,m_3}{a\,n}\,m^2\,\mathrm{U}_i^{(1)} = [\alpha', \alpha'_i] = \text{constant.}$$

Here the coefficient of S_i is greater than that of U_i in the ratio $\frac{m_1}{m_3} \div \frac{a\,n^3}{a'\,n'^3}$, or about a hundred millions to unity. The constant terms of $\mathrm{S}_i^{(1)}$ must therefore be to the periodic terms in at least this ratio. The coefficient of $\mathrm{U}_i^{(1)}$ may therefore be entirely neglected along side that of $\mathrm{S}_i^{(1)}$.

Again, the approximate value of the coefficient k is

$$k = \frac{m_3}{m_2}\frac{a^2}{a'^2} = \frac{1}{12{,}000{,}000}$$

the terms multiplied by this quantity may therefore be entirely neglected. The variations of the solar elements may therefore be determined entirely from the first of the pair of equations by a system of six equations of the form

$$S_1^{(1)}\frac{d\alpha'_1}{dt}+S_2^{(1)}\frac{d\alpha'_2}{dt}+\ \ldots\ +S_6^{(1)}\frac{d\alpha'_6}{dt}=\frac{m_4}{m_1}n'\,S^{(5)} \tag{43}$$

which will, in fact, be the usual equations obtained on the supposition that the masses of the Earth and Sun are combined in their center of gravity. This hypothesis is known to suffice even for the action of the Sun, *a fortiori* must it suffice for the disturbing action of a planet.

Let us next consider that in the pair of equations (34) and (35), α_j represents a lunar element. If we then express the first two equations (40) in the form

$$\left(\frac{d\,R}{d\,\alpha}\right)_1=\frac{m_2\,m_4}{a'}k\,U^{(7)}$$
$$\left(\frac{d\,R}{d\,\alpha}\right)_2=\frac{m_2\,m_4}{a'}k\,S^{(7)} \tag{44}$$

if, also, we substitute for the coefficients their values in (41) the pair of equations (34) and (35) assume the form

$$\frac{m_2\,m_3}{a\,n}\left(U_1^{(4)}\frac{d\,\alpha_1}{dt}+U_2^{(4)}\frac{d\,\alpha_2}{dt}+\ \ldots\ \right)+\frac{m_2\,m_3}{a\,n}m\left(U_1^{(3)}\frac{d\,\alpha'_1}{dt}+U_2^{(3)}\frac{d\,\alpha'_2}{dt}+\ \ldots\ \right)$$
$$=\frac{m_2\,m_4}{a'}k\,U^{(7)}$$

$$\frac{m_1\,m_2}{a'\,n'}k^2\left(S_1^{(4)}\frac{d\,\alpha_1}{dt}+S_2^{(4)}\frac{d\,\alpha_2}{dt}+\ \ldots\ \right)+\frac{m_1\,m_2}{a'\,n'}k\left(S_1^{(3)}\frac{d\,\alpha'_1}{dt}+S_2^{(3)}\frac{d\,\alpha'_2}{dt}+\ \ldots\ \right)$$
$$=\frac{m_2\,m_4}{a'}k\,S^{(7)}$$

The sum of these two equations forms the final equation for the variations of the lunar elements after the variations of the solar elements are determined from (43). If we multiply their terms by $a\,n\div m_2\,m_3$, putting for k its value

$$k=\frac{m_3}{m_2+m_3}\frac{a^2}{a'^2}$$

using the relations

$$a^3\ n^2\ =m_2+m_3$$
$$a'^3\,n'^2=m_1+m_2+m_3$$

and regarding as unity the ratio

$$m_1\div m_1+m_2+m_3$$

the equations become

the sign Σ indicating a summation of the terms which arise by writing x, y, z, x', y', z' in succession for x.

We also suppose that R becomes a function of the elements α_i, and of the time by the substitution of the values of the coordinates as functions of these quantities. The La Grangian equations for the variation of the elements then take the following form:

$$[\alpha_i\,\alpha_1]\frac{d\,\alpha_1}{d\,t}+[\alpha_i,\,\alpha_2]\frac{d\,\alpha_2}{d\,t}+\quad.\;.\;.\;.\;.\quad+[\alpha_i,\,\alpha_{12}]\frac{d\,\alpha_{12}}{d\,t}=\frac{d\,\mathrm{R}}{d\,\alpha_i} \tag{21}$$

$$(i=1;\;2;\;3;\;\;.\;.\;.\;.\;.\;.\;12)$$

In determining these La Grangian coefficients I make use of the known theorem that they are constants. This theorem rests, however, upon the supposition that the expressions for the coordinates actually satisfy the differential equations. As a matter of fact they are developed in series, so that it is impossible that this condition shall be rigorously fulfilled. It is therefore important to determine finally to what degree of approximation the constancy of the coefficients in question is assured. For our present purpose I have deemed it sufficient to develop the constant portions of the coefficients so far as the degree of approximation to which the solution is carried would admit.

This theorem of the constancy of the coefficients enables their values to be formed with much greater facility than if it were necessary to actually perform all the multiplications which enter into the equation (20).

In considering the general theory a yet farther simplification may be made. It is known that if we take six of the elements, which we may call

$$l_1,\;l_2,\;\;.\;.\;.\;.\;\;l_6$$

at pleasure, we can always substitute for the other six certain functions of them,

$$c_1,\;c_2\;\;.\;.\;.\;.\;\;c_6$$

such that the La Grangian coefficients shall reduce to zero in all the combinations except those six in which an l_i is coordinated with its corresponding c_i, in which case the combination will be reduced to unity. We shall make frequent use of this theorem in the course of the present investigation.

Such a selection of elements I shall hereafter call *canonical.*

§ 5.

THE PROBLEM OF THREE BODIES.

The preliminary step now is to express the six coordinates, x, y, z, x', y', z', in terms of twelve arbitrary constants, so as to satisfy the differential equations (16).

We recall that two processes of integration are necessary in the preliminary investigation, namely:

1. We integrate omitting Ω_1 in the equations (13), writing Ω_0 for $\Omega+\mathrm{R}$.
2. We include Ω_1, but omit R, as in the equations (16).

The first step involves no difficulty, since it gives rise only to the equations of elliptic motion—

(*a*) Of the Moon around the Earth;

(*b*) Of the center of gravity of the Earth and Moon around the Sun.

Putting in a general way

$$c_1,\ c_2,\ c_3,\ l_1,\ l_2,\ l_3$$

for a set of canonical elliptic elements pertaining to the Moon, and

$$c'_1,\ c'_2,\ c'_3,\ l'_1,\ l'_2,\ l'_3$$

for those pertaining to the Sun, we shall then have, as the result of putting $\Omega_1 = 0$

$$\begin{aligned} &x,\ y,\ \text{and } z = f(c_1,\ c_2,\ c_3,\ l_1,\ l_2,\ l_3,\ t) \text{ and} \\ &x',\ y',\ \text{and } z' = f(c'_1,\ c'_2,\ c'_3,\ l'_1,\ l'_2,\ l'_3,\ t) \end{aligned} \tag{22}$$

Next, to take account of Ω_1, we substitute in Ω_1, for x, y, z, x', y', z', their expressions (22). Then, using canonical elements, the form (21) will become

$$\begin{aligned} \frac{d c_i}{d t} &= \frac{d \Omega_1}{d l_i} & \frac{d l_i}{d t} &= -\frac{d \Omega_1}{d c_i} \\ & & & & (i = 1, 2, 3) \\ \frac{d c'_i}{d t} &= \frac{d \Omega_1}{d l'_i} & \frac{d l'_i}{d t} &= -\frac{d \Omega_1}{d c'_i} \end{aligned} \tag{23}$$

These equations are now to be integrated by successive approximations. We have first to integrate the equations for c and l on the supposition that c' and l' are constant. Next, by substituting these values of c and l in the equations for c' and l' and integrating, we obtain approximate expressions for these quantities. If necessary these new values of c' and l' can be employed to reintegrate the first set of equations, and so on.

The first integration has been performed by DELAUNAY in his great work *Théorie du mouvement de la Lune.* We may consider that, in this work, DELAUNAY has implicitly formed expressions c and l which substituted in (22) give values of x, y, and z that completely satisfy the equations

$$\mu_2 \frac{d^2 x}{d t^2} = \frac{d(\Omega_0 + \Omega_1)}{d x} \text{ etc.} \tag{24}$$

on the supposition that c'_i and l'_i are treated as constants. More exactly, if we put, in (14)

$$\begin{aligned} \Omega'_0 &= \frac{m_2 m_3}{r} \\ \Omega''_0 &= \frac{m_1(m_2 + m_3)}{r'} \end{aligned} \tag{25}$$

Delaunay's values of the coordinates, x, y, and z, of the Moon satisfy the differential equations

$$\mu_2 \frac{d^2 x}{d t^2} = \frac{d(\Omega'_0 + \Omega_1)}{d x}$$

with similar equations in y and z, provided that we regard Ω'_0 and Ω_1 as functions of x, y, z, x', y', z', before differentiating as to x, and afterwards substitute for x, y, z, their expressions in terms of c_i and l_i.

Since Ω''_0, when regarded as a function of the six coordinates, does not contain x, y, or z, we may add it to $\Omega'_0 + \Omega_1$ in the equations without altering the result. We thus find that the equations

$$\mu_2 \frac{d^2 x}{d t^2} = \frac{d \Omega}{d x}$$

will be identically satisfied with regard to all twelve constants in **Delaunay's** theory on the supposition above made.

It must be noted that c and l in these equations have not the same signification as in (22), but are now the arbitrary constants introduced by the integration of (23).

To complete the solution of the problem of three bodies we have next to integrate the second set of equations (23). We therefore substitute in Ω_1 the values of x, y, z, x', y', z' as functions of all twelve elements c, c', l, l'. But it is to be noted that at this stage we are to differentiate Ω_1 with respect to c' and l' only so far as these quantities are introduced through x', y', and z', because, from the mode of forming the equations (23) we have

$$\frac{d \Omega_1}{d c'} = \frac{d \Omega_1}{d x'} \frac{d x'}{d c'} + \frac{d \Omega_1}{d y'} \frac{d y'}{d c'} + \frac{d \Omega_1}{d z'} \frac{d z'}{d c'} \tag{26}$$

c' being any accented element.

Since Ω_1 contains the lunar elements, c and l, the integration of the equations

$$\frac{d c'_i}{d t} = \frac{d \Omega_1}{d l'_i} \text{ etc.} \tag{27}$$

will give c'_i and l'_i as functions of c_i, l_i and six new arbitrary constants, which we may again represent by c'_i and l'_i. By substituting these values of c'_i and l'_i in (22) our three equations

$$\mu \frac{d^2 x'}{d t^2} = \frac{d \Omega}{d x'} \text{ etc.}$$

will then be identically satisfied with respect to all twelve constants, provided that in representing the identity we first regard Ω as a function of the coordinates expressed in terms of the twelve constants.

By this process we have for the first time introduced the lunar elements c and l into the expressions for the Sun's coordinates. Hence, in strictness, the equations (23) will now no longer be satisfied with these new values of x', y', etc. It may be shown, however, that the error thus introduced is too small to affect the solution of the problem which now concerns us.

The result of integrating the equations in question may be expressed in the following way: By completing the solution of the problem of three bodies in the manner here proposed we shall implicitly obtain values of the six coordinates x, y, z, x' y', and z' as functions of the quantities

$$c_1, c_2, c_3;\ l_1, l_2, l_3;\ c'_1, c'_2, c'_3;\ l'_1, l'_2, l'_3;\ t \quad . \qquad (a)$$

which completely satisfy the differential equations

$$\mu_2 \frac{d^2 x}{d t^2} = \frac{d\Omega}{d x}$$

$$\vdots \qquad \vdots$$

$$\mu_1 \frac{d^2 x'}{d t^2} = \frac{d\Omega}{d x'}$$

$$\text{etc.} \qquad \text{etc.}$$

the two members becoming identical if, after differentiating Ω, we substitute for the coordinates their expressions in terms of the elements (a).

Regarding Ω as a function of the constants c, l, c', l', and of the time, we may indicate the satisfaction of the equations by expressing them in the form

$$\mu_2 \frac{d^2 x}{d t^2} = \Sigma \left\{ \frac{d\Omega}{d c_i} \frac{d c_i}{d x} + \frac{d\Omega}{d l_i} \frac{d l_i}{d x} + \frac{d\Omega}{d c'_i} \frac{d c'_i}{d x} + \frac{d\Omega}{d l'_i} \frac{d l'_i}{d x} \right\} \qquad (28)$$

with the five additional equations formed by writing successively y, z, x', y', z' in place of x in this equation, the seventy-two partial derivatives of c_i, l_i, etc., with respect to x, x', etc., being determined by the seventy-two equations each of one of the forms

$$\sum_{i=1}^{i=3} \left\{ \frac{d x}{d c_i} \frac{d c_i}{d x} + \frac{d x}{d l_i} \frac{d l_i}{d x} + \frac{d x}{d c'_i} \frac{d c'_i}{d x} + \frac{d x}{d l'_i} \frac{d l'_i}{d x} \right\} = 1$$

$$(29)$$

$$\sum_{i=1}^{i=3} \left\{ \frac{d x_1}{d c_i} \frac{d c_i}{d x} + \frac{d x_1}{d l_i} \frac{d l_i}{d x} + \frac{d x_1}{d c_i} \frac{d c_i}{d x} + \frac{d x_1}{d l'_i} \frac{d l'_i}{d x} \right\} = 0$$

the remaining seventy equations being formed by writing in succession x', x_1, x'_1, y, etc., in place of x or x_1 in the numerators found in these equations, and y, z, x', y', and z' for x in the denominators.

§ 6.

CONSIDERATIONS ON THE FORM AND ORDER OF MAGNITUDE OF THE EXPRESSIONS WHICH ENTER INTO THE PRECEDING THEORY.

The fifteen coefficients $[\alpha_i, \alpha_j]$ which enter into the equations (21) are each divisible into two parts, one of which is formed by differentiating the lunar coordinates x, y, and z, the other by differentiating the solar coordinates x', y', and z'. If we put

$$\begin{aligned}[\alpha_i\, \alpha_j]_1 = {} & \frac{dx}{d\alpha_i}\frac{dx_1}{d\alpha_j} + \frac{dy}{d\alpha_i}\frac{dy_1}{d\alpha_j} + \frac{dz}{d\alpha_i}\frac{dz_1}{d\alpha_j} \\ & - \frac{dx}{d\alpha_j}\frac{dx_1}{d\alpha_i} - \frac{dy}{d\alpha_j}\frac{dy_1}{d\alpha_i} - \frac{dz}{d\alpha_j}\frac{dz_1}{d\alpha_i}\end{aligned} \tag{30}$$

$$\begin{aligned}[\alpha_i\, \alpha_j]_2 = {} & \frac{dx'}{d\alpha_i}\frac{dx'_1}{d\alpha_j} + \frac{dy'}{d\alpha_i}\frac{dy'_1}{d\alpha_j} + \frac{dz'}{d\alpha_i}\frac{dz'_1}{d\alpha_j} \\ & - \frac{dx'}{d\alpha_j}\frac{dx'_1}{d\alpha_i} - \frac{dy'}{d\alpha_j}\frac{dy'_1}{d\alpha_i} - \frac{dz'}{d\alpha_j}\frac{dz'_1}{d\alpha_i}\end{aligned} \tag{31}$$

$$\left(\frac{d\mathrm{R}}{d\alpha_i}\right)_1 = \frac{d\mathrm{R}}{dx}\frac{dx}{d\alpha_i} + \frac{d\mathrm{R}}{dy}\frac{dy}{d\alpha_i} + \frac{d\mathrm{R}}{dz}\frac{dz}{d\alpha_i} \tag{32}$$

$$\left(\frac{d\mathrm{R}}{d\alpha_i}\right)_2 = \frac{d\mathrm{R}}{dx'}\frac{dx'}{d\alpha_i} + \frac{d\mathrm{R}}{dy'}\frac{dy'}{d\alpha_i} + \frac{d\mathrm{R}}{dz'}\frac{dz'}{d\alpha_i} \tag{33}$$

we may form by means of the expression (17) the two equations

$$\sum_{j=1}^{j=12} [\alpha_i, \alpha_j]_1 \frac{d\alpha_j}{dt} = \left(\frac{d\mathrm{R}}{d\alpha_i}\right)_1 \tag{34}$$

$$\sum_{j=1}^{j=12} [\alpha_i, \alpha_j]_2 \frac{d\alpha_j}{dt} = \left(\frac{d\mathrm{R}}{d\alpha_i}\right)_2 \tag{35}$$

and the sum of these two equations will make up the equation (21).

We may divide the twelve elements α into two classes, which may be designated *lunar* and *solar* elements. The first class comprise those which, when the Sun's disturbing force vanishes, merge into the elements of the Moon's elliptic motion around the Earth; the second, those which, when the mass of the Moon vanishes, merge into the elements of the Earth's elliptic motion around the Sun.

Let us next investigate the order of magnitude of these several coefficients and of $\left(\frac{d\mathrm{R}}{d\alpha}\right)_1$ and $\left(\frac{d\mathrm{R}}{d\alpha}\right)_2$. In the expressions (17) for the solar coordinates we are to distinguish two portions, one a purely elliptic part containing only the six elements of the elliptic motion of the Sun, the other part containing the terms added when we take into account the distance of the Moon as compared with that of the Sun. The first part arises from the integration of the differential equations for x', y', z' when only the part Ω_0 of the potential is considered, and is equal to the mean distance of the Sun, a', multiplied by purely numerical ratios.

In the expressions for x, y, and z, all the terms containing solar elements will be multiplied by $\frac{n'}{n}$.

We shall use the symbols S, S′, etc., to represent in the most general way the sum of a series of sines and cosines of angles, each multiplied into purely numerical ratios. These symbols will not twice represent the same quantity, and may, in special cases, reduce to purely numerical constants. If we put for brevity

$$\begin{aligned} m &= \frac{n'}{n} \\ k &= \mu \frac{a^2}{a'^2} \end{aligned} \tag{36}$$

the expressions of the form (17) and (18) for x, x_1, etc., and their derivatives may be put in the form

$$\begin{aligned} x &= a\,(\mathrm{S} + m\,\mathrm{S}') \\ x' &= a'\,(\mathrm{S}' + k\,\mathrm{S}) \\ x_1 &= \mu_2\,a\,n\,(\mathrm{S} + m\,\mathrm{S}') \\ x'_1 &= \mu_1\,a'\,n'\,(\mathrm{S}' + k\,\mathrm{S}) \\ &\text{etc.} \qquad \text{etc.} \end{aligned} \tag{37}$$

If, in forming the derivatives of these expressions as to the elements $\alpha_1 \ldots a_{12}$, we choose for the latter purely numerical quantities, which we may always do by replacing a line element by its logarithm, then the derivatives in question will be of the same form and the same order of magnitude with x, x', etc., themselves, or more exactly with the terms of those quantities which contain the elements. If, then, as before we distinguish the solar elements by accents, the order of magnitude of the largest terms of several derivatives will be as follow:

$$\begin{aligned} \frac{dx}{d\alpha} &= a\,\mathrm{S} \\ \frac{dx_1}{d\alpha} &= \mu_2\,a\,n\,\mathrm{S} \\ \frac{dx'}{d\alpha} &= k\,a'\,\mathrm{S} \\ \frac{dx'_1}{d\alpha} &= \mu_1\,k\,a'\,n'\,\mathrm{S} \end{aligned} \tag{38}$$

$$\left.\begin{aligned}
\frac{dx}{d\alpha'} &= m\,a\,S' \\
\frac{dx_1}{d\alpha'} &= \mu_2\,m\,a\,n\,S' \\
\frac{dx'}{d\alpha'} &= a'\,S' \\
\frac{dx'_1}{d\alpha'} &= \mu_1\,a'\,n'\,S'
\end{aligned}\right\} \tag{38}$$

The various classes of coefficients $[a_i, a_j]$ formed by (30) and (31) will therefore be of the following orders of magnitude:

$$\left.\begin{aligned}
[\alpha, \alpha_i]_1 &= \mu_2 a^2 n\,S \\
[\alpha', \alpha_i]_1 &= \mu_2 m\,a^2 n\,S \\
[\alpha, \alpha'_i]_1 &= \mu_2 m\,a^2 n\,S \\
[\alpha', \alpha'_i]_1 &= \mu_2 m^2 a^2 n\,S \\
[\alpha, \alpha_i]_2 &= \mu_1 k^2 a'^2 n'\,S \\
[\alpha, \alpha'_i]_2 &= \mu_1 k\,a'^2 n'\,S \\
[\alpha', \alpha_i]_2 &= \mu_1 k\,a'^2 n'\,S \\
[\alpha', \alpha'_i]_2 &= \mu_1 a'^2 n'\,S
\end{aligned}\right\} \tag{39}$$

Let us next revert to the perturbative function R as found in (14). We see that it is composed of two parts of very different magnitude. Considering all linear heliocentric coordinates as of the same order of magnitude, a', and the coordinates of the Moon as of the order of a, the order of magnitude of the two terms of R will be

$$R = \frac{m_4 \mu_1}{a'} S + \frac{m_4 \mu_2 a^2}{a'^3} S$$

or, if we use coordinates instead of mean distances, the orders of magnitude will be represented by

$$R = \frac{m_4 \mu_1}{x'} S + \frac{m_4 \mu_2 x^2}{x'^3} S$$

where the first term does not contain the Moon's coordinates.

From this equation we shall have, using the sign $=$ only to indicate identity of form and order of magnitude,

$$\frac{dR}{dx} = \frac{m_4 \mu_2 x}{x'^3} S = \frac{m_4 \mu_2 a}{a'^3} S$$

$$\frac{dR}{dx'} = \frac{m_4 \mu_1}{x'^2} S + \frac{m_4 \mu_2 x^2}{x'^4} S = \frac{m_4 \mu_1}{a'^2} S + \frac{m_4 \mu_2 a^2}{a'^4} S$$

and thence, from (32) and (33), having reference to the orders of magnitude of $\frac{dx}{d\alpha}$, etc.,

$$\begin{aligned}
\left(\frac{dR}{d\alpha}\right)_1 &= \frac{m_4\,\mu_2\,a^2}{a'^3}\,S = \frac{m_4\,\mu_1\,k}{a'}\,S \\
\left(\frac{dR}{d\alpha}\right)_2 &= \frac{m_4\,\mu_1 k}{a'}\,S \\
\left(\frac{dR}{d\alpha'}\right)_1 &= \frac{m_4\,\mu_2\,m\,a^2}{a'^3}\,S = \frac{m_4\,\mu_1\,m\,k}{a'}\,S \\
\left(\frac{dR}{d\alpha'}\right)_2 &= \frac{m_4\mu_1}{a'}\,S
\end{aligned} \tag{40}$$

The approximate values of μ_1 and μ_2 are

$$\mu_1 = m_2$$

$$\mu_2 = m_3$$

by which we shall replace them in R. The relations between the mean motions and mean distances give, approximately,

$$a^2\,n = \frac{m_2}{a\,n}$$

$$a'^2\,n' = \frac{m_1}{a'\,n'}$$

Substituting these values in the expressions (39) for the coefficients $[\alpha_1, \alpha_i]$, and representing by S_1, S_2, etc., U_1, U_2, etc., numerical quantities like those we have hitherto represented by the general symbol S, we have

$$\begin{aligned}
[\alpha, \ \alpha_i\,]_1 &= \frac{m_2\,m_3}{a\,n}\,U_i^{(4)} \\
[\alpha, \ \alpha'_i]_1 &= \frac{m_2\,m_3}{a\,n}\,m\,U_i^{(3)} \qquad (i = 1, 2 \ \ldots \ 6) \\
[\alpha', \alpha_i\,]_1 &= \frac{m_2\,m_3}{a\,n}\,m\,U_i^{(2)} \\
[\alpha', \alpha'_i]_1 &= \frac{m_2\,m_3}{a\,n}\,m^2\,U_i^{(1)} \\
[\alpha, \ \alpha_i\,]_2 &= \frac{m_1\,m_2}{a'\,n'}\,k^2\,S_i^{(4)} \\
[\alpha, \ \alpha'_i]_2 &= \frac{m_1\,m_2}{a'\,n'}\,k\,S_i^{(3)} \\
[\alpha', \alpha_i\,]_2 &= \frac{m_1\,m_2}{a'\,n'}\,k\,S_i^{(2)} \\
[\alpha', \alpha'_i]_2 &= \frac{m_1\,m_2}{a'\,n'}\,S_i^{(1)}
\end{aligned} \tag{41}$$

The expressions thus formed are to be substituted in the pair of general equations (34) and (35) which, it must be remembered, may belong to either of two classes, according as α_j there represents a solar (accented) element, or a lunar (unaccented) one. Let us consider the former case first. The right-hand members of these equations, taken from (40), may then be put in the form

$$\left(\frac{d\,\mathrm{R}}{d\,\alpha'}\right)_1 = m\,\frac{m_2\,m_4}{a'}\,k\,\mathrm{U}^{(5)} = \frac{m_3\,m_4}{a}\,m\,\frac{a^3}{a'^3}\,\mathrm{U}^{(5)}$$

$$\left(\frac{d\,\mathrm{R}}{d\,\alpha'}\right)_2 = \frac{m_2\,m_4}{a'}\left(\mathrm{S}^{(5)} + k\,\mathrm{S}^{(6)}\right)$$

The pair of equations then become

$$\frac{m_1\,m_2}{a'\,n'}\left\{\mathrm{S}_1^{(1)}\frac{d\,\alpha'_1}{d\,t} + \mathrm{S}_2^{(1)}\frac{d\,\alpha'_2}{d\,t} + \;.\;\;.\; + \mathrm{S}_6^{(1)}\frac{d\,\alpha'_6}{d\,t} + k\left(\mathrm{S}_1^{(2)}\frac{d\,\alpha_1}{d\,t} + \;.\;\;.\; + \mathrm{S}_6^{(2)}\frac{d\,\alpha_6}{d\,t}\right)\right\}$$

$$= \frac{m_2\,m_4}{a'}\left(\mathrm{S}^{(5)} + k\,\mathrm{S}^{(6)}\right)$$

$$\frac{m_2\,m_3}{a\,n}\left\{m^2\left(\mathrm{U}_1^{(1)}\frac{d\,\alpha'_1}{d\,t} + \;.\;\;.\; + \mathrm{U}_6^{(1)}\frac{d\,\alpha'_6}{d\,t}\right) + m\left(\mathrm{U}_1^{(2)}\frac{d\,\alpha_1}{d\,t} + \;.\;\;.\; + \mathrm{U}_1^{(2)}\frac{d\,\alpha_6}{d\,t}\right)\right\} \tag{42}$$

$$= \frac{m_3\,m_4}{a}\,m\,\frac{a^3}{a'^3}\,\mathrm{U}^{(5)}$$

The sum of these two equations gives one of the rigorous **La Grangian** equations for the variations of the elements, in which each coefficient of the differential of an element is a constant. The coefficient of $\frac{d\,\alpha'_i}{d\,t}$ in the sum is

$$\frac{m_1\,m_2}{a'\,n'}\,\mathrm{S}_i^{(1)} + \frac{m_2\,m_3}{a\,n}\,m^2\,\mathrm{U}_i^{(1)} = [\alpha',\,\alpha'_i] = \text{constant.}$$

Here the coefficient of S_i is greater than that of U_i in the ratio $\frac{m_1}{m_3} \div \frac{a\,n^3}{a'\,n'^3}$, or about a hundred millions to unity. The constant terms of $\mathrm{S}_i^{(1)}$ must therefore be to the periodic terms in at least this ratio. The coefficient of $\mathrm{U}_i^{(1)}$ may therefore be entirely neglected along side that of $\mathrm{S}_i^{(1)}$.

Again, the approximate value of the coefficient k is

$$k = \frac{m_3\,a^2}{m_2\,a'^2} = \frac{1}{12{,}000{,}000}$$

the terms multiplied by this quantity may therefore be entirely neglected. The variations of the solar elements may therefore be determined entirely from the first of the pair of equations by a system of six equations of the form

$$S_1^{(1)}\frac{d\alpha'_1}{dt}+S_2^{(1)}\frac{d\alpha'_2}{dt}+\quad\cdot\quad\cdot\quad\cdot\quad+S_6^{(1)}\frac{d\alpha'_6}{dt}=\frac{m_4}{m_1}n'\,S^{(5)} \tag{43}$$

which will, in fact, be the usual equations obtained on the supposition that the masses of the Earth and Sun are combined in their center of gravity. This hypothesis is known to suffice even for the action of the Sun, *a fortiori* must it suffice for the disturbing action of a planet.

Let us next consider that in the pair of equations (34) and (35), α_j represents a lunar element. If we then express the first two equations (40) in the form

$$\left.\begin{aligned}\left(\frac{d\,R}{d\,\alpha}\right)_1&=\frac{m_2\,m_4}{a'}k\,U^{(7)}\\ \left(\frac{d\,R}{d\,\alpha}\right)_2&=\frac{m_2\,m_4}{a'}k\,S^{(7)}\end{aligned}\right\} \tag{44}$$

if, also, we substitute for the coefficients their values in (41) the pair of equations (34) and (35) assume the form

$$\frac{m_2\,m_3}{a\,n}\left(U_1^{(4)}\frac{d\alpha_1}{dt}+U_2^{(4)}\frac{d\alpha_2}{dt}+\ \cdot\ \cdot\ \right)+\frac{m_2\,m_3}{a\,n}m\left(U_1^{(3)}\frac{d\alpha'_1}{dt}+U_2^{(3)}\frac{d\alpha'_2}{dt}+\ \cdot\ \cdot\ \right)$$

$$=\frac{m_2\,m_4}{a'}k\,U^{(7)}$$

$$\frac{m_1\,m_2}{a'\,n'}k^2\left(S_1^{(4)}\frac{d\alpha_1}{dt}+S_2^{(4)}\frac{d\alpha_2}{dt}+\ \cdot\ \cdot\ \right)+\frac{m_1\,m_2}{a'\,n'}k\left(S_1^{(3)}\frac{d\alpha'_1}{dt}+S_2^{(3)}\frac{d\alpha'_2}{dt}+\ \cdot\ \cdot\ \right)$$

$$=\frac{m_2\,m_4}{a'}k\,S^{(7)}$$

The sum of these two equations forms the final equation for the variations of the lunar elements after the variations of the solar elements are determined from (43). If we multiply their terms by $a\,n\div m_2\,m_3$, putting for k its value

$$k=\frac{m_3}{m_2+m_3}\,\frac{a^2}{a'^2}$$

using the relations

$$a^3\,n^2=m_2+m_3$$

$$a'^3\,n'^2=m_1+m_2+m_3$$

and regarding as unity the ratio

$$m_1\div m_1+m_2+m_3$$

the equations become

$$\mathrm{U}_1^{(4)}\frac{d\,\alpha_1}{d\,t}+\mathrm{U}_2^{(4)}\frac{d\,\alpha_2}{d\,t}+\quad.\quad.\quad.\quad+m\,\mathrm{U}_1^{(3)}\frac{d\,\alpha'_1}{d\,t}+m\,\mathrm{U}_2^{(3)}\frac{d\,\alpha'_2}{d\,t}+\quad.\quad.\quad.\quad=\frac{m_4\,n}{m_2+m_3}\frac{a^3}{a'^3}\mathrm{U}^{(7)}$$

$$m\,k\,(\mathrm{S}_1^{(4)}\,\frac{d\,\alpha_1}{d\,t}+\mathrm{S}_2^{(4)}\,\frac{d\,\alpha_2}{d\,t}+\quad.\quad.\quad)+m\,\mathrm{S}_1^{(3)}\frac{d\,\alpha'_1}{d\,t}+m\,\mathrm{S}_2^{(3)}\frac{d\,\alpha'_2}{d\,t}+\quad.\quad.\quad=\frac{m_4\,n}{m_2+m_3}\frac{a^3}{a'^3}\mathrm{S}^{(7)} \tag{45}$$

The sum of these two equations will give one of the **La Grangian** equations for the variations of the elements, in which the coefficients

$$\mathrm{U}_i^{(4)}+m\,k\,\mathrm{S}^4_{(i)},$$

and

$$m\,(\mathrm{U}_i^{(3)}+\mathrm{S}_i^{(3)})$$

are functions of the elements alone, and do not contain the time explicitly.

In these sums, however, the coefficient of $\mathrm{S}^{(4)}$ in the second equation is less than that of U^4 in the first in the ratio of $m\,k$ to 1, and since the periodic terms in the two coefficients destroy each other, the quantities $\mathrm{U}_1^{(4)}$, $\mathrm{U}_2^{(4)}$, etc., must be constant to their 150,000,000 part, and may therefore be regarded as absolute constants. In fact these coefficients $\mathrm{U}_i^{(4)}$ are the same as those which determine the variations of the lunar elements in what I have called the first method, and which is that employed by **Hill**.

Thus the final equations for the variations of the lunar elements may be written in the form.

$$\mathrm{U}_1^{(4)}\frac{d\,\alpha_1}{d\,t}+\mathrm{U}_2^{(4)}\frac{d\,\alpha_2}{d\,t}+\quad.\quad.\quad.\quad+\mathrm{U}_4^{(6)}\,\frac{d\,\alpha_6}{d\,t}=\frac{m_4\,n}{m_2+m_3}\frac{a^3}{a'^3}(\mathrm{U}^{(7)}+\mathrm{S}^{(7)})$$

$$-(\mathrm{U}_1^{(3)}+\mathrm{S}_1^{(3)})\,m\,\frac{d\alpha'_1}{d\,t}-(\mathrm{U}_2^{(3)}+\mathrm{S}_2^{(3)})\,m\,\frac{d\alpha'_2}{dt}-\text{etc.} \tag{46}$$

those of the solar elements being first derived from (43).

§ 7.

METHOD OF EXPRESSING THE COORDINATES IN TERMS OF THE ELEMENTS.

Our next step is to investigate the formation of the coefficients $[\alpha_i, \alpha_j]$ in (20). This presupposes that the rectangular coordinates x, y, z, x', y', z' are expressed in terms of the twelve arbitrary constants α_1 . . . α_{12} and of the time, in such a form as to be a complete solution of the differential equations (16). This requires that the coordinate planes and lines of reference be arbitrary, but no loss of astronomical rigor will be incurred if we suppose the fundamental plane to differ from that of the ecliptic at some epoch only by a quantity which may be treated as an infinitesimal.

The theory of **Delaunay** gives the Moon's longitude and latitude referred to the plane of the ecliptic and expressed in terms of the arbitrary constants

$$a,\ e,\ \gamma,\ a',\ e',\ h,\ g,\ l,\ h',\ g',\ l',$$

of which, however, one is superfluous.

The notation which I shall use is, however, different from that of DELAUNAY, the symbols given at the end of each line being those of DELAUNAY. I put

v; the Moon's true longitude, from an arbitrary origin . . . (V in DELAUNAY).
β; its latitude above the ecliptic . . . . (U " ").
l; its mean longitude . . . . ($h+g+l$ ").
g; its mean anomaly . . . . (l ").
θ; the longitude of its node on the ecliptic . . . . (h ").
γ; the sine of half of the inclination of the Moon's orbit to the ecliptic . . . . (γ ").

The corresponding accented quantities refer to the coordinates x', y', z' of the Sun.

DELAUNAY'S work now enables us to express the spherical coordinates of the Moon in the form

$$\begin{aligned} v &= l + \Sigma k \sin \mathrm{N}. \\ r &= \Sigma k' \cos \mathrm{N}. \\ \beta &= \Sigma c \sin \mathrm{N}'. \end{aligned} \tag{47}$$

The coefficients k, k' and c are functions of five independent elements

$$a,\ e,\ \gamma,\ a'\ e',$$

while N and N′ are linear functions with integral coefficients of four angles,

$$g,\ g',\ \omega,\ \omega'$$

ω and ω' being the mean distances of the lunar and solar perigees from the Moon's ascending node. Thus the coordinates of the Moon are expressed in terms of ten independent quantities, of which, however, one is only a constant additive to l or to v.

It is to be remarked that, so far as the problem of three bodies is concerned, the position of the ecliptic is to be regarded as invariable.

If we represent by ξ, η, ζ, the rectangular coordinates of the Moon referred to the ecliptic, and put, for brevity

$$\delta v = v - l = \Sigma k \sin \mathrm{N}$$

the expressions ξ, η and ζ may be written

$$\begin{aligned} \xi &= r \cos \beta \cos v = r \cos \beta \cos \delta v \cos l - r \cos \beta \sin \delta v \sin l \\ \eta &= r \cos \beta \sin v = r \cos \beta \cos \delta v \sin l + r \cos \beta \sin \delta v \cos l \\ \zeta &= r \sin \beta. \end{aligned} \tag{48}$$

Since β and δv are small quantities, the limit of whose values is little more than 0·1, the three quantities $r \cos \beta \cos \delta v$, $r \cos \beta \sin \delta v$ and $r \sin \beta$ each admit of being developed in a convergent series, as follows:

We first substitute for the sines and cosines of β and δv their developments

$$\sin \delta v = \delta v\left(1 - \frac{\delta v^2}{3!} + \frac{\delta v^4}{5!} - \frac{\delta v^6}{7!} + \text{etc.} \ldots\right)$$

$$\cos \delta v = 1 - \frac{\delta v^2}{2!} + \frac{\delta v^4}{4!} - \frac{\delta v^6}{6!} + \text{etc.}$$

$$\sin \beta = \beta\left(1 - \frac{\beta^2}{3!} + \frac{\beta^4}{5!} - \text{etc.} \ldots\right)$$

$$\cos \beta = 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \text{etc.} \ldots .$$

Forming the powers of δv and β and substituting for the powers and products of the sines and cosines their expressions in terms of the sines and cosines of the multiple angles, we see that the three quantities in question can be represented in the same general form with δv, r, and β, that is, in the form

$$r \cos \beta \cos \delta v = \Sigma h \cos \mathrm{N}$$

$$r \cos \beta \sin \delta v = \Sigma h' \sin \mathrm{N}$$

$$r \sin \beta = \Sigma c' \sin \mathrm{N}'$$

h, h' and N being of the same form as k and N in the values of v, β and r in (47). Thus the values of ξ, η, and ζ in (48) assume the form

$$\begin{aligned} \xi &= \Sigma k \cos(l + \mathrm{N}) \\ \eta &= \Sigma k \sin(l + \mathrm{N}) \\ \zeta &= \Sigma c \sin \mathrm{N}' \end{aligned} \qquad (49)$$

where

$$k = h \pm h'$$

and N is still a linear function of the four quantities

$$g, g', \omega, \omega'.$$

The corresponding coordinates of the Sun may be expressed in the same general form, which we may write

$$\begin{aligned} \xi' &= \Sigma k' \cos(l' + \mathrm{N}) \\ \eta' &= \Sigma k' \sin(l' + \mathrm{N}) \\ \zeta' &= \Sigma c' \sin \mathrm{N}' \end{aligned} \qquad (50)$$

It is however to be remarked that, in the case of the Sun, the principal values of k' will correspond to cases in which N depends on g' alone, and that when g, ω or ω' enter into N or N$'$, the values of k' and c' will contain the minute factor k of § 6.

Let us now suppose the positions of the Sun and Moon referred to any system of rectangular coordinates whatever. If we represent by

$$\mathrm{a},\ \mathrm{b},\ \mathrm{c};\ \mathrm{a}',\ \mathrm{b}',\ \mathrm{c}';\ \mathrm{a}'',\ \mathrm{b}'',\ \mathrm{c}'',$$

the nine direction cosines for transforming ξ, η, ζ into x, y, z, and if we put

θ', the ascending node of the ecliptic on the plane of XY counted from the axis of X,

γ', the sine of half the inclination of these planes,

τ, the distance of the same node from the axis of Ξ,

we shall have

$$\begin{aligned}
\mathrm{a} &= (1-\gamma'^2)\cos(\theta'-\tau)+\gamma'^2\cos(\theta'+\tau)\\
\mathrm{b} &= -(1-\gamma'^2)\sin(\theta'-\tau)+\gamma'^2\sin(\theta'+\tau)\\
\mathrm{c} &= 2\gamma'\sqrt{1-\gamma'^2}\sin\theta'\\
\mathrm{a}' &= (1-\gamma'^2)\sin(\theta'-\tau)+\gamma'^2\sin(\theta'+\tau)\\
\mathrm{b}' &= (1-\gamma'^2)\cos(\theta'-\tau)-\gamma'^2\cos(\theta'+\tau)\\
\mathrm{c}' &= -2\gamma'\sqrt{1-\gamma'^2}\cos\theta'\\
\mathrm{a}'' &= -2\gamma'\sqrt{1-\gamma'^2}\sin\tau\\
\mathrm{b}'' &= 2\gamma'\sqrt{1-\gamma'^2}\cos\tau\\
\mathrm{c}'' &= 1-2\gamma'^2
\end{aligned}\tag{51}$$

By means of the values of ξ, η, and ζ given in (49), and the above values of a, b, etc., we find for the general rectangular coordinates

$$\begin{aligned}
x = {} & \Sigma k(1-\gamma'^2)\cos(\theta'-\tau+l+\mathrm{N})\\
& +\Sigma k\gamma'^2\cos(\theta'+\tau-l-\mathrm{N})\\
& -\Sigma c\gamma'\sqrt{1-\gamma'^2}\cos(\theta'+\mathrm{N}')\\
& +\Sigma c\gamma'\sqrt{1-\gamma'^2}\cos(\theta'-\mathrm{N}')\\
y = {} & \Sigma k(1-\gamma'^2)\sin(\theta'-\tau+l+\mathrm{N})\\
& +\Sigma k\gamma'^2\sin(\theta'+\tau-l-\mathrm{N})\\
& -\Sigma c\gamma'\sqrt{1-\gamma'^2}\sin(\theta'+\mathrm{N}')\\
& +\Sigma c\gamma'\sqrt{1-\gamma'^2}\sin(\theta'-\mathrm{N}')
\end{aligned}\tag{52}$$

$$
\begin{aligned}
z &= 2\,\Sigma\, k\, \gamma' \sqrt{1-\gamma'^2}\sin(l-\tau+\mathrm{N}) \\
&\quad +\Sigma\,(1-2\,\gamma'^2)\, c \sin \mathrm{N}' \\
x' &= \Sigma\, k'\,(1-\gamma'^2)\cos(\theta'-\tau+l'+\mathrm{N}) \\
&\quad +\Sigma\, k'\,\gamma'^2 \cos(\theta'+\tau-l'-\mathrm{N}) \\
&\quad -\Sigma\, c'\,\gamma' \sqrt{1-\gamma'^2}\cos(\theta'+\mathrm{N}') \\
&\quad +\Sigma\, c'\,\gamma' \sqrt{1-\gamma'^2}\cos(\theta'-\mathrm{N}') \\
y' &= \Sigma\, k'\,(1-\gamma'^2)\sin(\theta'-\tau+l'+\mathrm{N}) \\
&\quad +\Sigma\, k'\,\gamma'^2 \sin(\theta'+\tau-l'-\mathrm{N}) \\
&\quad -\Sigma\, c'\,\gamma' \sqrt{1-\gamma'^2}\sin(\theta'+\mathrm{N}') \\
&\quad +\Sigma\, c'\,\gamma' \sqrt{1-\gamma'^2}\sin(\theta'-\mathrm{N}') \\
z' &= 2\,\Sigma\, k'\,\gamma' \sqrt{1-\gamma'^2}\sin(l'-\tau+\mathrm{N}) \\
&\quad +\Sigma\,(1-2\,\gamma'^2)\, c' \sin \mathrm{N}'
\end{aligned}
\qquad (52)
$$

Examining these equations, we see that the coefficients of the varying angles are now functions of the six constants

$$a,\ e,\ \gamma,\ a',\ e',\gamma'$$

and that the angles under the signs *sin* and *cos* may be expressed in terms of the time and six other arbitrary constants. For these angles contain only θ', $\tau-l$, N, and N'. The two latter can be expressed as a function of the four quantities already mentioned, $\tau-l$ need contain but one constant, and thus, with θ', we have the required number of six.

To reduce the preceding values of x, y, and z to a common form exhibiting these six arbitrary constants, let us represent the four varying angles of which N and N' are functions by λ, λ', ϖ, ϖ'. These quantities are not to be considered as having a perfectly defined geometrical signification, being only certain analytical expressions which substituted in the equations (47) will give numerical values of the Moon's coordinates. But, we may consider that they represent

λ, the mean distance of the Moon from her node on the ecliptic.

ϖ, the distance of the perigee from the same node.

λ', ϖ', the same quantities for the Sun, counted from the same node.

We shall now express all the quantities found under the sign *sin* or *cos* in x, y, and z in terms of the six arbitrary quantities

$$\varepsilon,\ \pi,\ \theta,\ \varepsilon',\ \pi',\ \theta'$$

the first five of which are determined by the conditions

$$\begin{aligned}
&\varepsilon = \theta' + l - \tau \\
&\pi = \theta + \omega \\
&\theta = \varepsilon - \lambda \\
&\varepsilon' = \lambda' - \lambda + \varepsilon, = \theta' + l' - \tau \\
&\pi' = \theta + \omega'
\end{aligned}$$

They are therefore introduced by putting in the expressions for x, y, and z,

$$\left.\begin{aligned}
l - \tau &= \varepsilon - \theta' \\
l' - \tau &= \varepsilon' - \theta' \\
\lambda &= \varepsilon - \theta \\
\lambda' &= \varepsilon' - \theta \\
\omega &= \pi - \theta \\
\omega' &= \pi' - \theta
\end{aligned}\right\} \qquad (53)$$

The expressions for x, y, and z thus take the form

$$\begin{aligned}
x &= \Sigma k \cos\,(i\,\varepsilon + i^{\rm I}\pi + i^{\rm II}\theta + i^{\rm III}\varepsilon' + i^{\rm IV}\pi' + i^{\rm V}\theta') \\
y &= \Sigma k \sin\,(i\,\varepsilon + i^{\rm I}\pi + i^{\rm II}\theta + i^{\rm III}\varepsilon' + i^{\rm IV}\pi' + i^{\rm V}\theta') \qquad (54) \\
z &= \Sigma c \sin\,(j\,\varepsilon + j^{\rm I}\pi + j^{\rm II}\theta + j^{\rm III}\varepsilon' + j^{\rm IV}\pi' + j^{\rm V}\theta')
\end{aligned}$$

the coefficients i and j being positive or negative integers satisfying the conditions

$$\begin{aligned}
i + i^{\rm I} + i^{\rm II} + i^{\rm III} + i^{\rm IV} + i^{\rm V} &= 1 \\
j + j^{\rm I} + j^{\rm II} + j^{\rm III} + j^{\rm IV} + j^{\rm V} &= 0
\end{aligned}$$

and k and c being functions of the six quantities a, a', e, e', γ, γ'.

The values of x', y', and z' in (52) can be reduced to the same form, so that it is unnecessary to write them.

§ 8.

METHOD OF FORMING THE LA GRANGIAN COEFFICIENTS.

It will be seen by what precedes that our problem requires as a preliminary work the complete solution of the problem of three bodies by obtaining not only the relative coordinates of the Moon and the Earth, but the coordinates of the Sun relative to the center of gravity of the Earth and Moon, which has not yet been done. Instead of effecting the solution of this preliminary problem at the present stage, I shall proceed on the supposition that this part of the work is done, and proceed with the investigation growing out of it, in order that more exact information may thus be obtained as to the degree of approximation to which the solution must be carried.

It is shown in my paper of 1875, on the General Integrals of Planetary Motion,* that when the solution in question is carried on by successive approximations, the expressions for such coordinates as x, y, z, x', y', z', in terms of the twelve arbitrary constants which were required in the solution of the problem of three bodies, could always be reduced to a general form of the kind found in the last section, namely

$$x \text{ or } x' = \Sigma\, k \cos \mathrm{N}$$
$$y \text{ or } y' = \Sigma\, k \sin \mathrm{N}$$
$$z \text{ or } z' = \Sigma\, c \sin \mathrm{N}'$$

in which each value of N and of N′ is of the form

$$\mathrm{N} = i_1\, \lambda_1 + i_2\, \lambda_2 + \quad . \quad . \quad . \quad + i_6\, \lambda_6$$
$$\mathrm{N}' = j_1\, \lambda_1 + j_2\, \lambda_2 + \quad . \quad . \quad . \quad + j_6\, \lambda_6$$

i_1, i_2, . . i_6, j_1 . . . j_6, being integers satisfying the conditions

$$i_1 + i_2 + \quad . \quad . \quad + i_6 = 0$$
$$j_1 + j_2 + \quad . \quad . \quad + j_3 = 1$$

and λ_1 . . . λ_6 expressions of the form

$$\lambda_i = l_i + b_i\, t$$

l_1, l_2, . . . l_6 being arbitrary constants, and b_1, b_2, . . . b_6 as well as the various values of k and of c, functions of six other arbitrary constants, which, if used as in the theory of DELAUNAY, may be called

$$a,\ e,\ \gamma,\ a',\ e',\ \gamma'$$

The symbol Σ indicates a summation extended to all combinations of integral values of the indices i_1, i_2, etc.

As special values of the six constants l we may take those which in the ordinary theory are designated as follows:

θ, the longitude of the Moon's node, or DELAUNAY's h;

π, the longitude of the Moon's perigee, or DELAUNAY's $h+g$;

l, the Moon's mean longitude, or DELAUNAY's $h+g+l$;

θ', π', l', the corresponding quantities relating to the Sun.

It is to be remarked that the assignment of the usual astronomical designation to these quantities does not adequately define them, and may therefore be misleading. No precise definition is possible except by their analytic expressions as integrals, each of them being regarded as a function of the coordinates, their first derivatives, and the time. This definition might, however, be considered as replaced by any complete solution expressing the values of the coordinates in terms of the arbitrary constants and of the time. We shall therefore simply suppose that expressions of the preceding form for the rectangular coordinates are obtained in terms of two sets of six arbitrary constants each, which satisfy the differential equations for the case of three bodies to the necessary degree of approximation.

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*Smithsonian Contributions to Knowledge, Vol. XXI.

The danger of error in assigning geometrical significations to the elements will be seen by considering the special quantity which Delaunay has called the longitude of the Moon's node. His entire lunar theory is constructed on the supposition that the plane of the ecliptic is invariable, an hypothesis which is sufficiently correct when the action of the planets is omitted, so far, at least, as any secular variation is concerned. But when we consider the complete problem of three bodies we shall find that there are two planes, either of which may be called that of the ecliptic. One of these is the plane of maximum areal velocity, which is really invariable. The other is the plane determined at each instant by the relative motion of the Sun and of the center of gravity of the Earth and Moon, which plane is variable.

Since all general equations between physical quantities are necessarily homogeneous when each quantity is expressed in units of fundamental or arbitrary elements, it will generally conduce to clearness to write them in such a way that this homogeneousness shall always be manifest. In physics the three fundamental elements whose units are arbitrary are mass, length, and time. But in astronomy it will generally be simpler, as it is more common, to regard the unit of mass as a derived one, and determined by the condition that two bodies of unit mass shall attract each other with unit force when at unit distance. If we represent by the letters M, L, and T the dimensions mass, length, and time respectively, the equation for M to which we are thus led is

$$M = L^3\, T^{-2}$$

In a theory like the present it will conduce to clearness to write all equations in such a way that the homogenity shall be apparent. Since N and N′ are angles, and therefore pure numbers, it follows that λ, l, and $b\,t$ are also pure numbers. Hence b is of dimensions T^{-1}, as are also n and n', the mean motions. Since the coordinates are lines all the coefficients k are of the form

$$a \text{ or } a' \times \text{a pure number}$$

while $b_1, b_2 \;\ldots\; b_6$ are each of the form $n\,s$, or $n'\,s$, s being a pure number.

Hence each conjugate variable of the kind

$$x_1 = \mu_2 \frac{dx}{dt}$$

is of the dimensions $M\,L\,T^{-1}$ or $L^4\,T^{-3}$ according as we do or do not retain M, and may be written in the form

$$m\,a\,n \times \text{a pure number.}$$

The six pairs of conjugate variables from which the La Grangian coefficients are to be formed may therefore be written in the form

$$\begin{aligned} x &= \Sigma\, k_i \cos N_i & \qquad x_1 &= -\Sigma\, k'_j \sin N_j \\ y &= \Sigma\, k_i \sin N_i & \qquad y_1 &= \Sigma\, k'_j \cos N_j \\ z &= \Sigma\, c_i \sin N'_i & \qquad z_1 &= \Sigma\, c'_j \cos N'_j \end{aligned} \qquad (55)$$

where the indices i and j are used to distinguish the terms belonging to the various linear combinations which make up the values of N. We have, in each case,

$$k' = \mu b k; \quad c' = \mu b' c$$

μ representing either μ_1 or μ_2, according as the coordinates are those of the Moon or of the Sun, and b and b' being the coefficients of t in N and N', so that in general

$$\begin{aligned} b &= i_1 b_1 + i_2 b_2 + \quad . \quad . \quad . \quad + i_6 b_6 \\ b' &= j_1 b_1 + j_2 b_2 + \quad . \quad . \quad . \quad + j_6 b_6 \end{aligned} \tag{56}$$

Now, let α and β be any two elements for which we are to form the LA GRANGIAN coefficient $[\alpha, \beta]$. Putting no restriction as to the class to which either of these elements may belong, we have from (55) the following values of the derivatives from which $[\alpha_i, \beta]$ is to be formed

$$\frac{dx}{d\alpha} = \Sigma\left(\frac{dk_i}{d\alpha}\cos N_i - k_i \frac{dN_i}{d\alpha}\sin N_i\right)$$

$$\frac{dx_1}{d\beta} = \Sigma\left(-\frac{dk'_j}{d\beta}\sin N_j - k'_j \frac{dN_j}{d\beta}\cos N_j\right)$$

$$\frac{dx}{d\beta} = \Sigma\left(\frac{dk_i}{d\beta}\cos N_i - k_i \frac{dN_i}{d\beta}\sin N_i\right)$$

$$\frac{dx_1}{d\alpha} = \Sigma\left(-\frac{dk'_j}{d\alpha}\sin N_j - k'_j \frac{dN_j}{d\alpha}\cos N_j\right)$$

$$\frac{dy}{d\alpha} = \Sigma\left(\frac{dk_i}{d\alpha}\sin N_i + k_i \frac{dN_i}{d\alpha}\cos N_i\right)$$

$$\frac{dy_1}{d\beta} = \Sigma\left(\frac{dk'_j}{d\beta}\cos N_j - k'_j \frac{dN_j}{d\beta}\sin N_j\right)$$

$$\frac{dy}{d\beta} = \Sigma\left(\frac{dk_i}{d\beta}\sin N_i + k_i \frac{dN_i}{d\beta}\cos N_i\right)$$

$$\frac{dy_1}{d\alpha} = \Sigma\left(\frac{dk'_j}{d\alpha}\cos N_j - k'_j \frac{dN_j}{d\alpha}\sin N_j\right)$$

$$\frac{dz}{d\alpha} = \Sigma\left(\frac{dc_i}{d\alpha}\sin N'_i + c_i \frac{dN'_i}{d\alpha}\cos N'_i\right)$$

$$\frac{dz_1}{d\beta} = \Sigma\left(\frac{dc'_j}{d\beta}\cos N'_j - c'_j \frac{dN'_j}{d\beta}\sin N'_j\right)$$

$$\frac{dz}{d\beta} = \Sigma\left(\frac{dc_i}{d\beta}\sin N'_i + c_i \frac{dN'_i}{d\beta}\cos N'_i\right)$$

$$\frac{dz_1}{d\alpha} = \Sigma\left(\frac{dc'_j}{d\alpha}\cos N'_j - c'_j \frac{dN'_j}{d\alpha}\sin N'_j\right)$$

Substituting these values in the expressions of the form (30), p. 122, and putting for brevity

$$[\varphi, \theta] = \frac{d\varphi}{d\alpha}\frac{d\theta}{d\beta} - \frac{d\varphi}{d\beta}\frac{d\theta}{d\alpha}$$

we find

$$\begin{aligned}
[\alpha, \beta]_1 = {} & \Sigma^2 \{[k_i, k'_j] + k_i k'_j [N_i, N_j]\} \sin (N_i - N_j) \\
& + \Sigma^2 \{k_i [N_i, k'_j] - k'_j [k_i, N_j]\} \cos (N_i - N_j) \\
& + \frac{1}{2} \Sigma^2 \{[c_i, c'_j] + c_i c'_j [N'_i, N'_j]\} \sin (N'_i - N'_j) \\
& + \frac{1}{2} \Sigma^2 \{[c_i, c'_j] - c_i c'_j [N'_i, N'_j]\} \sin (N'_i + N'_j) \qquad (57) \\
& + \frac{1}{2} \Sigma^2 \{c_i [N'_i, c'_j] - c'_j [c_i, N'_j]\} \cos (N'_i - N'_j) \\
& + \frac{1}{2} \Sigma^2 \{c_i [N'_i, c'_j] + c'_j [c_i, N'_j]\} \cos (N'_i + N'_j)
\end{aligned}$$

It will be understood that the two indices i, j, have each the same system of values, and their combination under the sign Σ^2 indicates that each value of N is to be combined with all the other values, itself included.

In order to find the function $[\alpha, \beta]$ of which we are in quest, it is necessary to omit all the periodic terms in (57). Substituting for N_i and N_j their values $f_i + b_i t$ and $f_j + b_j t$, we see that all the terms will be periodic, unless we have either

$$b_i - b_j = 0$$

or

$$b'_i \pm b'_j = 0$$

The values of b_i and b_j being from (22)

$$\begin{aligned}
b_i &= i_{1,i} b_1 + i_{2,i} b_2 + \quad . \quad . \quad . \quad + i_{6,i} b_6 \\
b_j &= i_{1,j} b_1 + i_{2,j} b_2 + \quad . \quad . \quad . \quad + i_{6,j} b_6 \\
b'_i &= j_{1,i} b_1 + j_{2,i} b_2 + \quad . \quad . \quad . \quad + j_{6,i} b_6 \\
b'_j &= j_{1,j} b_1 + j_{2,j} b_2 + \quad . \quad . \quad . \quad + j_{6,j} b_6
\end{aligned}$$

it is necessary that we have either

$$(i_{1,i} - i_{1,j}) b_1 + (i_{2,i} - i_{2,j}) b_2 + \quad . \quad . \quad . \quad + (i_{6,i} - i_{6,j}) b_6 = 0$$

or

$$(j_{1,i} \pm j_{1,j}) b_1 + (j_{2,i} \pm j_{2,j}) b_2 + \quad . \quad . \quad . \quad + (j_{6,i} \pm j_{6,j}) b_6 = 0$$

But b_1 . . . b_6 being incommensurable quantities, these equations can not be satisfied except we have

$$i_{1,i}=\quad i_{i,j}\qquad i_{2,i}=\quad i_{2,j}\quad .\quad .\quad .\quad i_{6,i}=\quad i_{6,j}$$

$$j_{1,i}=\mp j_{2,j}\qquad j_{2,i}=\mp j_{2,j}\quad .\quad .\quad .\quad j_{6,i}=\pm j_{6,j}$$

from which follows

$$f_i=f_j\qquad\qquad \mathrm{N}_i=\mp\mathrm{N}_j$$

Therefore we retain in the equation (57) only those terms which satisfy the condition

$$i=j$$

We have, therefore, omitting the index i,

$$\begin{aligned}[\alpha,\beta]_1= &\ \Sigma\{k[\mathrm{N},k']-k'[k,\mathrm{N}]\}\\ &+\frac{1}{2}\Sigma\{c[\mathrm{N}',c']-c'[c,\mathrm{N}']\}\end{aligned}$$

or, placing for k' its value $\mu_2 b\,k$, and $\mu_2 b'\,c$ for c',

$$\begin{aligned}[\alpha,\beta]_1= &\ \Sigma\left[\frac{d(\mu_2 b k^2)}{d\beta}\frac{d\mathrm{N}}{d\alpha}-\frac{d(\mu_2 b\,k^2)}{d\alpha}\frac{d\mathrm{N}}{d\beta}\right]\\ &+\frac{1}{2}\Sigma\left[\frac{d(\mu_2 b' c^2)}{d\beta}\frac{d\mathrm{N}'}{d\alpha}-\frac{d(\mu_2 b' c^2)}{d\alpha}\frac{d\mathrm{N}'}{d\beta}\right]\end{aligned}$$

Omitting in $\frac{d\,\mathrm{N}}{d\,\alpha}$ the terms which contain t as a factor, we then have

$$\begin{aligned}[\alpha,\beta]_1=\mu_2\,\Sigma &\left[\frac{d\,f}{d\,\alpha}\frac{d\,(b\,k^2)}{d\,\beta}-\frac{d\,f}{d\,\beta}\,\frac{d\,(b\,k^2)}{d\,\alpha}\right]\\ \frac{1}{2}\,\mu_2\,\Sigma &\left[\frac{df'}{d\,\alpha}\,\frac{d\,(b'\,c^2)}{d\,\beta}-\frac{d\,f'}{d\,\beta}\,\frac{d\,(b'\,c^2)}{d\,\alpha}\right]\end{aligned}\tag{58}$$

It will be remembered that α and β here represent any pair of the twelve arbitrary constants which enter into the expressions (55) for x, y, and z. These constants are divisible into two classes.

Class (1) comprises the elements

$$a,\ e,\ \gamma,\ a',\ e',\ \gamma',$$

which enter into, b, b', k, and c, but not into f.

Class (2) comprises the elements

$$l_1,\ l_2,\quad .\quad .\quad .\quad .\quad l_6$$

which enter into f and f', but not into b, b', c, or k.

It is evident from the form of the preceding expression for $[\alpha\ \beta]$ that it will vanish whenever the elements α and β belong to the same class. We have therefore only to consider the combinations in which they belong to different classes, of which the number is 18.

Let us now conceive the general quantities $\lambda_1 \quad . \quad . \quad . \quad \lambda_6$ to be replaced by the six elements

$$\varepsilon,\ \pi,\ \theta,\ \varepsilon',\ \pi',\ \theta',$$

already defined, and let us put

$$n;\ \pi_1, \quad . \quad . \quad . \quad . \quad \theta_1',$$

for the motions of the corresponding elements in unit time. Each value of b or b' may then be written

$$b = i\,n + i^{\text{I}}\,\pi_1 + i^{\text{II}}\,\theta_1 + i^{\text{III}}\,n' + i^{\text{IV}}\,\pi'_1 + i^{\text{V}}\,\theta'_1$$

$$b' = j\,n + j^{\text{I}}\,\pi_1 + j^{\text{II}}\,\theta_1 + j^{\text{III}}\,n' + j^{\text{IV}}\,\pi'_1 + j^{\text{V}}\,\theta'_1$$

We then have

$$\frac{d\,f}{d\,\varepsilon} = \frac{d\,\mathrm{N}}{d\,\varepsilon} = i \qquad \frac{d\,f'}{d\,\varepsilon} = \frac{d\,\mathrm{N}'}{d\,\varepsilon} = j$$

$$\frac{d\,f}{d\,\pi} = \frac{d\,\mathrm{N}}{d\,\pi} = i' \qquad \frac{d\,f'}{d\,\pi} = \frac{d\,\mathrm{N}'}{d\,\pi} = j'$$

$$. \quad . \qquad\qquad . \quad .$$

$$. \quad . \qquad\qquad . \quad .$$

$$\frac{d\,f}{d\,\theta'} = \frac{d\,\mathrm{N}}{d\,\theta'} = i^{\text{V}} \qquad \frac{d\,f'}{d\,\theta'} = \frac{d\,\mathrm{N}'}{d\,\theta'} = j^{\text{V}}$$

We have therefore, supposing β to represent in succession the six elements of the second class and restoring the sign Σ,

$$\begin{aligned}
[\alpha,\,\varepsilon]_2 &= -\mu_2 \Sigma\, i \frac{d\,(b\,k^2)}{d\,\alpha} - \frac{1}{2}\mu_2 \Sigma\, j \frac{d\,(b'\,c^2)}{d\,\alpha} \\
[\alpha,\pi]_2 &= -\mu_2 \Sigma\, i^{\text{I}} \frac{d\,(b\,k^2)}{d\,\alpha} - \frac{1}{2}\mu_2 \Sigma\, j^{\text{I}} \frac{d\,(b'\,c^2)}{d\,\alpha} \qquad (59) \\
. \quad . &\qquad\quad . \quad . \qquad\qquad . \\
. \quad . &\qquad\quad . \quad . \\
[\alpha,\,\theta]_2 &= -\mu_2 \Sigma\, i^{\text{V}} \frac{d\,(b\,k^2)}{d\,\alpha} - \frac{1}{2}\mu_2 \Sigma\, j^{\text{V}} \frac{d\,(b'\,c^2)}{d\,\alpha}
\end{aligned}$$

The sign Σ here refers to the several terms which enter into the expresssions (54). The entire number of sensible coefficients $[\alpha,\,\beta]_1$ is now reduced to thirty-six.

To complete the required expressions we still require the values of $[\alpha\ \beta]_2$. The rigorous values of x', y', and z' can be developed in precisely the same way with those of x, y, and z. As we have seen all the terms containing a, e, γ, ε, π, or θ will contain the minute factor k in (36), but for the sake of generality we shall suppose them all retained. If we put

$$\begin{aligned}
x' &= \Sigma\, k \cos(\mathrm{I}\,\varepsilon + \mathrm{I}^{\text{I}}\,\pi + \mathrm{I}^{\text{II}}\theta + \mathrm{I}^{\text{III}}\,\varepsilon' + \mathrm{I}^{\text{IV}}\,\pi' + \mathrm{I}^{\text{V}}\,\theta') \\
y' &= \Sigma\, k \sin(\mathrm{I}\,\varepsilon + \mathrm{I}^{\text{I}}\,\pi + \mathrm{I}^{\text{II}}\theta + \mathrm{I}^{\text{III}}\,\varepsilon' + \mathrm{I}^{\text{IV}}\,\pi' + \mathrm{I}^{\text{V}}\,\theta') \\
z' &= \Sigma\, c \sin(\mathrm{J}\,\varepsilon + \mathrm{J}^{\text{I}}\,\pi + \mathrm{J}^{\text{II}}\theta + \mathrm{J}^{\text{III}}\,\varepsilon' + \mathrm{J}^{\text{IV}}\,\pi' + \mathrm{J}^{\text{V}}\,\theta') \qquad (60) \\
\mathrm{B} &= \mathrm{I}\,n + \mathrm{I}^{\text{I}}\,\pi_1 + \mathrm{I}^{\text{II}}\,\theta_1 + \mathrm{I}^{\text{III}}n' + \mathrm{I}^{\text{IV}}\,\pi'_1 + \mathrm{I}^{\text{V}}\,\theta'_1 \\
\mathrm{B}' &= \mathrm{J}\,n + \mathrm{J}^{\text{I}}\,\pi_1 + \mathrm{J}^{\text{II}}\,\theta_1 + \mathrm{J}^{\text{III}}n' + \mathrm{J}^{\text{IV}}\,\pi'_1 + \mathrm{J}^{\text{V}}\,\theta'_1
\end{aligned}$$

we have as in the case of the lunar coordinates

$$
\begin{aligned}
[\alpha,\ \varepsilon]_2 &= -\mu_1 \Sigma\, \mathrm{I}\ \frac{d(\mathrm{BK}^2)}{d\,\alpha} - \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}\ \frac{d\,(\mathrm{B}'\,\mathrm{C}^2)}{d\,\alpha} \\
[\alpha,\ \pi]_2 &= -\mu_1 \Sigma\, \mathrm{I}^{\mathrm{I}}\ \frac{d(\mathrm{BK}^2)}{d\,\alpha} - \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{I}}\ \frac{d\,(\mathrm{B}'\,\mathrm{C}^2)}{d\,\alpha} \qquad (61) \\
&\cdot \qquad \cdot \qquad \cdot \\
&\cdot \qquad \cdot \qquad \cdot \\
&\cdot \qquad \cdot \qquad \cdot \\
[\alpha,\ \theta']_2 &= -\mu_1 \Sigma\, \mathrm{I}^{\mathrm{V}}\, \frac{d(\mathrm{BK}^2)}{d\,\alpha} - \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{V}}\, \frac{d\,(\mathrm{B}'\,\mathrm{C}^2)}{d\,\alpha}
\end{aligned}
$$

Finally, we have for the complete La Grangian coefficient

$$[\alpha,\ \beta] = [\alpha,\ \beta]_1 + [\alpha,\ \beta]_2$$

We conclude that the values of the La Grangian coefficients may be obtained as follows. Let us determine the six quantities k_ε, k_π, k_θ, k_ε', k_π', k_θ' from the equations:

$$
\begin{aligned}
k_\varepsilon &= \mu_1 \Sigma\, \mathrm{I}\ \mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}\ \ \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i\ \ b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j\ \ b'\,c^2 \\
k_\pi &= \mu_1 \Sigma\, \mathrm{I}^{\mathrm{I}}\ \mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{I}}\ \ \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i^{\mathrm{I}}\ b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j^{\mathrm{I}}\ b'\,c^2 \\
k_\theta &= \mu_1 \Sigma\, \mathrm{I}^{\mathrm{II}}\,\mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{II}}\ \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i^{\mathrm{II}}\, b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j^{\mathrm{II}}\, b'\,c^2 \qquad (62) \\
k_\varepsilon' &= \mu_1 \Sigma\, \mathrm{I}^{\mathrm{III}}\,\mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{III}}\ \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i^{\mathrm{III}}\, b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j^{\mathrm{III}}\, b'\,c^2 \\
k_\pi' &= \mu_1 \Sigma\, \mathrm{I}^{\mathrm{IV}}\,\mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{IV}}\ \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i^{\mathrm{IV}}\, b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j^{\mathrm{IV}}\, b'\,c^2 \\
k_\theta' &= \mu_1 \Sigma\, \mathrm{I}^{\mathrm{V}}\ \mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{V}}\ \ \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i^{\mathrm{V}}\, b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j^{\mathrm{V}}\, b'\,c^2
\end{aligned}
$$

We shall then have

$$
\begin{aligned}
&[a,\ \varepsilon] = -\frac{d\,k_\varepsilon}{d\,a}; \qquad [a,\ \pi] = -\frac{d\,k_\pi}{d\,a}\ \ \cdot\ \cdot\ \cdot \\
&[e,\ \varepsilon] = -\frac{d\,k_\varepsilon}{d\,e}; \qquad [e,\ \pi] = -\frac{d\,k_\pi}{d\,e}\ \ \cdot\ \cdot\ \cdot \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \qquad (63) \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&[\gamma'\ \varepsilon] = -\frac{d\,k_\varepsilon}{d\,\gamma'}; \qquad [\gamma'\ \pi] = -\frac{d\,k_\pi}{d\,\gamma'}\ .
\end{aligned}
$$

§ 9.

REDUCTION OF THE EQUATIONS TO THE CANONICAL FORM.

Let us now resume the equations (21), supposing the symbols α to represent in succession the twelve arbitrary elements

$$a, e, \gamma, a', e', \gamma', \varepsilon_0, \pi_0, \theta_0, \varepsilon'_0, \pi'_0, \theta'_0$$

where the six elements ε_0, π_0 . . . . . represent the constant parts of ε, π . . . . Observing that all pairs of constants, both members of which are found either among the first six or the last six of these constants, will make the combination (α_i, α) vanish, and that we have

$$\frac{dR}{d\varepsilon_0}=\frac{dR}{d\varepsilon} \qquad \frac{dR}{d\pi_0}=\frac{dR}{d\pi}, \text{ etc.}$$

the equations (21) by substituting the values of the coefficients in (63) become

$$\begin{aligned}
&\frac{dk_\varepsilon}{da}\frac{da}{dt}+\frac{dk_\varepsilon}{de}\frac{de}{dt}+\frac{dk_\varepsilon}{d\gamma}\frac{d\gamma}{dt}+\frac{dk_\varepsilon}{da'}\frac{da'}{dt}+\frac{dk_\varepsilon}{de'}\frac{de'}{dt}+\frac{dk_\varepsilon}{d\gamma'}\frac{d\gamma'}{dt}=\frac{dR}{d\varepsilon}\\
&\frac{dk_\pi}{da}\frac{da}{dt}+\frac{dk_\pi}{de}\frac{de}{dt}+\frac{dk_\pi}{d\gamma}\frac{d\gamma}{dt}+\frac{dk_\pi}{da'}\frac{da'}{dt}+\frac{dk_\pi}{de'}\frac{de'}{dt}+\frac{dk_\pi}{d\gamma'}\frac{d\gamma'}{dt}=\frac{dR}{d\pi}\\
&\frac{dk_\theta}{da}\frac{da}{dt}+\frac{dk_\theta}{de}\frac{de}{dt}+\frac{dk_\theta}{d\gamma}\frac{d\gamma}{dt}+\frac{dk_\theta}{da'}\frac{da'}{dt}+\frac{dk_\theta}{de'}\frac{de'}{dt}+\frac{dk_\theta}{d\gamma'}\frac{d\gamma'}{dt}=\frac{dR}{d\theta}\\
&\frac{dk'_\varepsilon}{da}\frac{da}{dt}+\frac{dk'_\varepsilon}{de}\frac{de}{dt}+\frac{dk'_\varepsilon}{d\gamma}\frac{d\gamma}{dt}+\frac{dk'_\varepsilon}{da'}\frac{da'}{dt}+\frac{dk'_\varepsilon}{de'}\frac{de'}{dt}+\frac{dk'_\varepsilon}{d\gamma'}\frac{d\gamma'}{dt}=\frac{dR}{d\varepsilon'}\\
&\frac{dk'_\pi}{da}\frac{da}{dt}+\frac{dk'_\pi}{de}\frac{de}{dt}+\frac{dk'_\pi}{d\gamma}\frac{d\gamma}{dt}+\frac{dk'_\pi}{da'}\frac{da'}{dt}+\frac{dk'_\pi}{de'}\frac{de'}{dt}+\frac{dk'_\pi}{d\gamma'}\frac{d\gamma'}{dt}=\frac{dR}{d\pi'}\\
&\frac{dk'_\theta}{da}\frac{da}{dt}+\frac{dk'_\theta}{de}\frac{de}{dt}+\frac{dk'_\theta}{d\gamma}\frac{d\gamma}{dt}+\frac{dk'_\theta}{da'}\frac{da'}{dt}+\frac{dk'_\theta}{de'}\frac{de'}{dt}+\frac{dk'_\theta}{d\gamma'}\frac{d\gamma'}{dt}=\frac{dR}{d\theta'}\\
&\frac{dk_\varepsilon}{da}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{da}\frac{d\pi_0}{dt}+\frac{dk_\theta}{da}\frac{d\theta_0}{dt}+\frac{dk'_\varepsilon}{da}\frac{d\varepsilon'_0}{dt}+\frac{dk'_\pi}{da}\frac{d\pi'_0}{dt}+\frac{dk'_\theta}{da}\frac{d\theta'_0}{dt}=-\frac{dR}{da}\\
&\frac{dk_\varepsilon}{de}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{de}\frac{d\pi_0}{dt}+\frac{dk_\theta}{de}\frac{d\theta_0}{dt}+\frac{dk'_\varepsilon}{de}\frac{d\varepsilon'_0}{dt}+\frac{dk'_\pi}{de}\frac{d\pi'_0}{dt}+\frac{dk'_\theta}{de}\frac{d\theta'_0}{dt}=-\frac{dR}{de}\\
&\frac{dk_\varepsilon}{d\gamma}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{d\gamma}\frac{d\pi_0}{dt}+\frac{dk_\theta}{d\gamma}\frac{d\theta_0}{dt}+\frac{dk'_\varepsilon}{d\gamma}\frac{d\varepsilon'_0}{dt}+\frac{dk'_\pi}{d\gamma}\frac{d\pi'_0}{dt}+\frac{dk'_\theta}{d\gamma}\frac{d\theta'_0}{dt}=-\frac{dR}{d\gamma}\\
&\frac{dk_\varepsilon}{da'}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{da'}\frac{d\pi_0}{dt}+\frac{dk_\theta}{da'}\frac{d\theta_0}{dt}+\frac{dk'_\varepsilon}{da'}\frac{d\varepsilon'_0}{dt}+\frac{dk'_\pi}{da'}\frac{d\pi'_0}{dt}+\frac{dk'_\theta}{da'}\frac{d\theta'_0}{dt}=-\frac{dR}{da'}\\
&\frac{dk_\varepsilon}{de'}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{de'}\frac{d\pi_0}{dt}+\frac{dk_\theta}{de'}\frac{d\theta_0}{dt}+\frac{dk'_\varepsilon}{de'}\frac{d\varepsilon'_0}{dt}+\frac{dk'_\pi}{de'}\frac{d\pi'_0}{dt}+\frac{dk'_\theta}{de'}\frac{d\theta'_0}{dt}=-\frac{dR}{de'}\\
&\frac{dk_\varepsilon}{d\gamma'}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{d\gamma'}\frac{d\pi_0}{dt}+\frac{dk_\theta}{d\gamma'}\frac{d\theta_0}{dt}+\frac{dk'_\varepsilon}{d\gamma'}\frac{d\varepsilon'_0}{dt}+\frac{dk'_\pi}{d\gamma'}\frac{d\pi'_0}{dt}+\frac{dk'_\theta}{d\gamma'}\frac{d\theta'_0}{dt}=-\frac{dR}{d\gamma'}
\end{aligned}\tag{64}$$

Since k_ϵ, k_π, etc., are functions of the six elements a, e, γ, a', e', γ' the left hand member of the first of these equations is equal to $\frac{d\,k_\epsilon}{d\,t}$ that of the second to $\frac{d\,k_\pi}{d\,t}$ and so on through the first six equations. The latter thus reduce to the six left hand equations in (65) below.

The quantities k_ϵ, k_π, etc., being six in number, we may conceive the six elements a, e, γ, a', e', γ', and R to be expressed in terms of these six new elements. If, then, we multiply the seventh of the preceding equations by $\frac{d\,a}{d\,k_\epsilon}$ the eighth by $\frac{d\,e}{d\,k_\epsilon}$ and so on to the twelfth, which we multiply by $\frac{d\,\gamma'}{d\,k_\epsilon}$ and notice that the theory of differentiation of inverse functions gives us

$$\frac{d\,k_\epsilon}{d\,a}\frac{d\,a}{d\,k_\epsilon}+\frac{d\,k_\epsilon}{d\,e}\frac{d\,e}{d\,k_\epsilon}+\frac{d\,k_\epsilon}{d\,\gamma}\frac{d\,\gamma}{d\,k_\epsilon}+\frac{d\,k_\epsilon}{d\,a'}\frac{d\,a'}{d\,k_\epsilon}+\frac{d\,k_\epsilon}{d\,e'}\frac{d\,e'}{d\,k_\epsilon}+\frac{d\,k_\epsilon}{d\,\gamma'}\frac{d\,\gamma'}{d\,k_\epsilon}=1$$

$$\frac{d\,k_\pi}{d\,a}\frac{d\,a}{d\,k_\epsilon}+\frac{d\,k_\pi}{d\,e}\frac{d\,e}{d\,k_\epsilon}+\frac{d\,k_\pi}{d\,\gamma}\frac{d\,\gamma}{d\,k_\epsilon}+\frac{d\,k_\pi}{d\,a'}\frac{d\,a'}{d\,k_\epsilon}+\frac{d\,k_\pi}{d\,e'}\frac{d\,e'}{d\,k_\epsilon}+\frac{d\,k_\pi}{d\,\gamma'}\frac{d\,\gamma'}{d\,k_\epsilon}=0$$

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with similar equations for every combination of two k elements, we find

$$\frac{d\,\epsilon_0}{d\,t}=-\frac{d\,\mathrm{R}}{d\,k_\epsilon}$$

In the same way we obtain

$$\frac{d\,\pi_0}{d\,t}=-\frac{d\,\mathrm{R}}{d\,k_\pi}$$

etc. etc.

The equations (64) may therefore be reduced to the canonical form

$$\begin{aligned}
\frac{d\,k_\epsilon}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\epsilon} & \frac{d\,\epsilon_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k_\epsilon}\\
\frac{d\,k_\pi}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\pi} & \frac{d\,\pi_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k_\pi}\\
\frac{d\,k_\theta}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\theta} & \frac{d\,\theta_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k_\theta}\\
\frac{d\,k'_\epsilon}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\epsilon'} & \frac{d\,\epsilon'_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k'_\epsilon}\\
\frac{d\,k'_\pi}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\pi'} & \frac{d\,\pi'_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k'_\pi}\\
\frac{d\,k'_\theta}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\theta'} & \frac{d\,\theta'_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k'_\theta}
\end{aligned} \qquad (65)$$

We may regard these as the rigorous differential equations for the solution of the problem of four bodies after the solution of the problem is effected for the case of three bodies.

In deriving the equations (64) from (21), we have limited ourselves by no hypothesis respecting the signification of the elements ε, π, etc., which appear under the signs *sin* and *cos*. We have only supposed the variable angles to be linear functions of six variables each of the form $a_0+b\,t$, a_0 being an arbitrary constant and b a function of the six arbitrary constants which form the coefficients k. We may therefore substitute for any of the quantities ε, π, θ, ε', π', θ', any linear function of other independent quantities completely determined in terms of ε, π, etc., by equations of the first degree. For every such system of new variables we shall have a system of canonical equations corresponding to (65). The number of these canonical forms is infinite.

The transformation in question can indeed be performed immediately on the equations (65). Let us determine six new variables $l_1, l_2, l_3, l_4, l_5, l_6$, by the conditions

$$\left.\begin{aligned}
p_1\,l_1+p_2\,l_2+p_3\,l_3+p_4\,l_4+p_5\,l_5+p_6\,l_6&=\varepsilon_0\\
q_1\,l_1+q_2\,l_2+q_3\,l_3+q_4\,l_4+q_5\,l_5+q_6\,l_6&=\pi_0\\
r_1\,l_1+r_2\,l_2+r_3\,l_3+r_4\,l_4+r_5\,l_5+r_6\,l_6&=\theta_0\\
s_1\,l_1+s_2\,l_2+s_3\,l_3+s_4\,l_4+s_5\,l_5+s_6\,l_6&=\varepsilon'_0\\
t_1\,l_1+t_2\,l_2+t_3\,l_3+t_4\,l_4+t_5\,l_5+t_6\,l_6&=\pi'_0\\
u_1\,l_1+u_2\,l_2+u_3\,l_3+u_4\,l_4+u_5\,l_5+u_6\,l_6&=\theta'_0
\end{aligned}\right\} \qquad (66)$$

p, q, r, etc., being any constants whatever of which the determinant does not vanish. Let us put also

$$\begin{aligned}
c_1&=p_1\,k_\varepsilon+q_1\,k_\pi+r_1\,k_\theta+s_1\,k_\varepsilon'+t_1\,k_\pi'+u_1\,k_\theta'\\
c_2&=p_2\,k_\varepsilon+q_2\,k_\pi+r_2\,k_\theta+s_2\,k_\varepsilon'+t_2\,k_\pi'+u_2\,k_\theta'\\
c_6&=p_6\,k_\varepsilon+q\,k_\pi+r_6\,k_\theta+s_6\,k_\varepsilon'+t_6\,k_\pi'+u_6\,k_\theta'
\end{aligned} \qquad (67)$$

Multiply the first of the equations (65) on the left by p_1, the second by q_1, etc., and add the six products thus formed. The sum reduces by (67) to

$$\frac{d\,c_1}{d\,t}=p_1\frac{d\,\mathrm{R}}{d\,\varepsilon_0}+q_1\frac{d\,\mathrm{R}}{d\,\pi_0}+r_1\frac{d\,\mathrm{R}}{d\,\theta_0}+s_1\frac{d\,\mathrm{R}}{d\,\varepsilon'_0}+t_1\frac{d\,\mathrm{R}}{d\pi'_0}+u_1\frac{d\,\mathrm{R}}{d\theta'_0}$$

The equations (66) give

$$p_1=\frac{d\,\varepsilon_0}{d\,l_1};\ \ q_1=\frac{d\,\pi_0}{d\,l_1}\ \text{etc.}$$

Substituting these values of p_1, q_1, etc., in the last equation it becomes

$$\frac{d\,c_1}{d\,t}=\frac{d\,\mathrm{R}}{d\,l_1}$$

We have in the same way

$$\frac{d\,c_2}{d\,t}=\frac{d\,\mathrm{R}}{d\,l_2};\ \ \frac{d\,c_3}{d\,t}=\frac{d\,\mathrm{R}}{d\,l_3}\ \text{etc.}$$

To obtain the values of $\frac{d\,l}{d\,t}$ let us suppose that we solve the equations (66) with respect to l_1, l_2, etc., so as to obtain the values of these quantities in terms of ε_0, π_0, θ_0, etc. These values will be of the form

$$l_1 = P_1\,\varepsilon_0 + Q_1\,\pi_0 + R_1\,\theta_0 + S_1\,\varepsilon'_0 + T_1\,\pi'_0 + U_1\,\theta'_0$$

$$l_2 = P_2\,\varepsilon_0 + Q_2\,\pi_0 + R_2\,\theta_0 + S_2\,\varepsilon'_0 + T_2\,\pi'_0 + U_2\,\theta'_0$$

. . . .

. . . .

$$l_6 = P_6\,\varepsilon_0 + Q_6\,\pi_0 + R_6\,\theta_0 + S_6\,\varepsilon'_0 + T_6\,\pi'_0 + U_6\,\theta'_0$$

P, Q, etc., being functions of p, q, etc., which are readily formed by the theory of determinants. From the theorem that any two determinants are identical when the horizontal lines of the one are identical with the vertical lines of the other it follows that the values of k_ε, k_π, etc., derived from (67) will be

$$k_\varepsilon = P_1\,c_1 + P_2\,c_2 + P_3\,c_3 + P_4\,c_4 + P_5\,c_5 + P_6\,c_6$$

$$k_\pi = Q_1\,c_1 + Q_2\,c_2 + Q_3\,c_3 + Q_4\,c_4 + Q_5\,c_5 + Q_6\,c_6$$

. . . .

. . . .

$$k_\theta' = U_1\,c_1 + U_2\,c_2 + U_3\,c_3 + U_4\,c_4 + U_5\,c_5 + U_6\,c_6$$

Let us now multiply the first of the right hand equations (65) by P_i, the second by Q_i, etc., and add the products. We shall have

$$\frac{d\,l_i}{d\,t} = -\frac{d\,R}{d\,c_i}$$

We may therefore replace the equations (65) by the more general ones

$$\left.\begin{aligned}
\frac{d\,c_1}{d\,t} &= \frac{d\,R}{d\,l_1} & \qquad \frac{d\,l_1}{d\,t} &= -\frac{d\,R}{d\,c_1}\\
\frac{d\,c_2}{d\,t} &= \frac{d\,R}{d\,l_2} & \qquad \frac{d\,l_2}{d\,t} &= -\frac{d\,R}{d\,c_2}\\
\frac{d\,c_3}{d\,t} &= \frac{d\,R}{d\,l_3} & \qquad \frac{d\,l_3}{d\,t} &= -\frac{d\,R}{d\,c_3}\\
\frac{d\,c_4}{d\,t} &= \frac{d\,R}{d\,l_4} & \qquad \frac{d\,l_4}{d\,t} &= -\frac{d\,R}{d\,c_4}\\
\frac{d\,c_5}{d\,t} &= \frac{d\,R}{d\,l_5} & \qquad \frac{d\,l_5}{d\,t} &= -\frac{d\,R}{d\,c_5}\\
\frac{d\,c_6}{d\,t} &= \frac{d\,R}{d\,l_6}\cdot & \qquad \frac{d\,l_6}{d\,t} &= -\frac{d\,R}{d\,c_6}
\end{aligned}\right\} \qquad (68)$$

Let us, for the sake of clearness, recapitulate the process by which we have arrived at these equations. Considering first only the mutual action of the Sun, the Earth, and the Moon, we obtain the rectangular coordinates of the Moon relatively to the Earth (x, y, z) and those of the Sun relatively to the center of gravity of the Earth and Moon (x', y', z') in the form

$$\left.\begin{aligned} x &= \Sigma\, k \cos(\lambda + b\,t) \\ y &= \Sigma\, k \sin(\lambda + b\,t) \\ z &= \Sigma\, c \sin(\lambda' + b'\,t) \\[1ex] x' &= \Sigma\, \mathrm{K} \cos(A + \mathrm{B}\,t) \\ y' &= \Sigma\, \mathrm{K} \sin(A + \mathrm{B}\,t) \\ z' &= \Sigma\, \mathrm{C} \sin(A' + \mathrm{B}'\,t) \end{aligned}\right\} \qquad (69)$$

k, c, b, b', K, C, B, and B$'$ being functions of the six constants

$$a,\ e,\ \gamma,\ a',\ e',\ \gamma'$$

and λ, λ', A, and A' being linear functions of six other constants

$$l_1, l_2, l_3, l_4, l_5, l_6,$$

which fix the mean longitudes of the bodies, and the positions of the lines of apsides and of the nodes by any system of linear equations like (66) which we choose to adopt. Of the infinite series of values of λ, λ', A, A', represented by the sign Σ each will be of the form

$$\lambda \text{ or } A = \mathrm{I}_1\, l_1 + \mathrm{I}_2\, l_2 + \mathrm{I}_3\, l_3 + \mathrm{I}_4\, l_4 + \mathrm{I}_5\, l_5 + \mathrm{I}_6\, l_6$$

(2) For each term of x or each value of k we form the product

$$\frac{m_2\, m_3}{m_2 + m_3}\, b\, k^2$$

which, for the moment, we represent by k'.

For each term of z, or each value of c we form the product

$$\frac{1}{2}\, \frac{m_2\, m_3}{m_2 + m_3}\, b'\, c^2$$

which we may represent by c'.

For each term of x', or each value of A we form the product

$$\frac{m_1\, (m_2 + m_3)}{m_1 + m_2 + m_3}\, \mathrm{B}\, \mathrm{K}_2 = \mathrm{K}'$$

For each term of z', or each value of A' form the product

$$\frac{1}{2}\, \frac{m_1 (m_2 + m_3)}{m_1 + m_2 + m_3}\, \mathrm{B}'\, \mathrm{C}^2 = \mathrm{C}'$$

(3) Multiply each k' thus formed by the corresponding value of i_1, or the coefficient of l_1 in the corresponding value of λ; each c' by the coefficient of l_1 in the corresponding value of λ'; each K$'$ by the coefficient of l_1 in the corresponding value of A, and each C$'$ by the coefficient of l_1 in the corresponding value of A'. Represent the sum of all the products thus formed by c_1.

In the same way form the value of c_2 from the coefficients of l_2, and so on to c_6. These six values of c_i will then be functions of the six elements

$$a, e, \gamma, a', e', \gamma',$$

and may therefore be conceived to replace them in the expressions for the coordinates.

(4) The disturbing function R being a function of the coordinates x, y, z, x', y', z', let these quantities be replaced by their values in term of the elements $c_1, c_2, \ldots c_6$, $l_1 l_2 \ldots l_6$. Then determine the values of these elements as variables by the integration of the equations

$$\frac{d\,c_i}{d\,t}=\frac{d\,\mathrm{R}}{d\,l_i} \qquad \frac{d\,l_i}{d\,t}=-\frac{d\,\mathrm{R}}{d\,c_i}$$

The values of c_i, l_i thus obtained being substituted in the expressions for the coordinates we shall have the values of the latter under the influence of the disturbing force.

§ 10.

TRANSFORMATION OF THE PRECEDING EQUATIONS.

The values of the elements deduced from the equations (65) or (68) are to be substituted in (69) to obtain the coordinates of the Moon relatively to the Earth under the influence of the attraction of the planet. If we take the equations in their literal acceptation, we shall find terms in which the coefficients of the sines or cosines of the varying angles have the time as a factor. These terms are introduced in the following way: We have assumed that in the disturbed motion the coordinates are expressed in terms of the elements in the same form as in the undisturbed motion, only that the elements are variable. In the undisturbed motion each of the variables

$$\varepsilon, \pi, \theta, \varepsilon', \pi'$$

is expressed in the form

$$l+b\,t,$$

l being an arbitrary constant of the second class, and b a function of the arbitrary constants of the first class. In the disturbed motion the derivative of this expression will be

$$\frac{d\,l}{d\,t}+t\frac{d\,b}{d\,t}+b$$

The expression $\frac{d\,l}{d\,t}$ being formed by taking the derivative of R with respect to quantities which are multiplied by the time in the varying angles, will contain the factor t outside the signs sin and cos.

We shall now show that these terms are destroyed by the term $t\,\frac{d\,b}{d\,t}$. Let us, as before, represent the angles by which replace ε, π, etc., in the general equations (68) by the general symbols

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$$

where

$$\lambda_1 = l_1 + b_1 t$$

$$\lambda_2 = l_2 + b_2 t$$

etc. etc.

We then have for any one λ and the corresponding c

$$\frac{d c_i}{d t} = \frac{d \mathrm{R}}{d \lambda_i} \qquad \frac{d l_i}{d t} = -\frac{d \mathrm{R}}{d c_i} \tag{70}$$

and

$$\frac{d \lambda_i}{d t} = \frac{d l_i}{d t} + t \frac{d b_i}{d t} + b \tag{71}$$

The terms of R in question are of the form

$$\mathrm{R} = h \cos(i_1 \lambda_1 + i_2 \lambda_2 + i_3 \lambda_3 + i_4 \lambda_4 + i_5 \lambda_5 + i_6 \lambda_6) = h \cos \mathrm{N}.$$

Consider any one of them. Since λ_1, λ_2, etc., are functions of c_i, through b_i, the value of $\frac{d \mathrm{R}}{d c_i}$ will contain the terms

$$\frac{d \mathrm{R}}{d \lambda_1}\frac{d \lambda_1}{d c_i} + \frac{d \mathrm{R}}{d \lambda_2}\frac{d \lambda_1}{d c_i} + \quad . \quad .$$

But

$$\frac{d \mathrm{R}}{d \lambda_1} = -i_1 h \sin \mathrm{N}; \quad \frac{d \mathrm{R}}{d \lambda_2} = -i_2 h \sin \mathrm{N}, \text{ etc.}$$

and

$$\frac{d \lambda_1}{d c_i} = t \frac{d b_1}{d c_i}; \quad \frac{d \lambda_2}{d c_i} = t \frac{d b_2}{d c_i}, \text{ etc.}$$

Therefore the value of $-\frac{d \mathrm{R}}{d c_i}$, and consequently of $\frac{d l_i}{d t}$, will contain the term

$$\left(i_1 \frac{d b_1}{d c_i} + i_2 \frac{d b_2}{d c_i} + \quad . \quad + i_6 \frac{d b_6}{d c_i}\right) t h \sin \mathrm{N}$$

To find the second term of $\frac{d \lambda_i}{d t}$ in (71) we have

$$\frac{d b_i}{d t} = \frac{d b_i}{d c_1}\frac{d c_1}{d t} + \frac{d b_i}{d c_2}\frac{d c_2}{d t} + \quad . \quad . \quad . \quad + \frac{d b_i}{d c_6}\frac{d c_6}{d t}$$

Substituting for $\frac{d c_1}{d t}$, $\frac{d c_2}{d t}$, etc., their values in (70), and for $\frac{d \mathrm{R}}{d \lambda}$ the expression just found, we have

$$t \frac{d b_i}{d t} = -\left(i_1 \frac{d b_i}{d c_1} + i_2 \frac{d b_i}{d c_2} + \quad . \quad . \quad . \quad + i_6 \frac{d b_i}{d c_6}\right) t h \sin \mathrm{N}$$

Each term of R will therefore produce in $\frac{d\lambda_i}{dt}$ the term

$$\left\{ i_1 \left(\frac{db_1}{dc_i} - \frac{db_i}{dc_1} \right) + i_2 \left(\frac{db_2}{dc_i} - \frac{db_i}{dc_2} \right) + \quad . \quad . \quad . \quad \right\} h\, t \sin \mathrm{N}.$$

It is known that the values of b are so related to those of c that for all combinations of the indices i and j we have

$$\frac{db_i}{dc_j} - \frac{db_j}{dc_i} = 0 \tag{72}$$

The general proof of this theorem is found in my paper on the General Integrals of Planetary Motion, so that it is not necessary to repeat it here

It appears that we may omit the term $t\frac{db}{dt}$ in the value of $\frac{d\lambda}{dt}$ in the disturbed orbit; provided that in taking the derivatives with respect to c, we suppose the values of b under the signs sin and cos, to be invariable, and vary only the coefficients k. We then have in the disturbed orbit

$$\frac{d\lambda_i}{dt} = \frac{dl_i}{dt} + b_i$$

$$\lambda_i = l_i + \int b_i\, dt$$

This result may be expressed in the elegant form adopted by DELAUNAY in his theory of the Moon. Taking λ_1, λ_2, etc., as the variables instead of l_1, l_2, etc., the above equations reduce the forms (68) to the following:—

$$\frac{dc_i}{dt} = \frac{d\mathrm{R}}{d\lambda_i} \qquad \frac{d\lambda_i}{dt} = -\frac{d\mathrm{R}}{dc_i} + b_i \tag{73}$$

Consider the differential function

$$b_1\, dc_1 + b_2\, dc_2 + \quad . \quad . \quad . \quad + b_6\, dc_6$$

The equations (72) show that this expression is an exact differential of a certain function of c_1, c_2, etc., which we may represent by Θ. If, then, we find θ by integrating this differential, we have

$$b_i = \frac{d\Theta}{dc_i}$$

Substituting this value of b_i in the above expressions for $\frac{d\lambda_i}{dt}$, and putting

$$\mathrm{R}' = \mathrm{R} - \Theta$$

the equations which give c_i and λ_i become

$$\frac{dc_i}{dt} = \frac{d\mathrm{R}'}{d\lambda_i} \qquad \frac{d\lambda_i}{dt} = -\frac{d\mathrm{R}'}{dc_i} \tag{74}$$

The fact that the mean motions of the Sun and Moon and the secular variation of the nodes and apsides of the solar and lunar orbits under the influence of the mutual attractions of the three bodies can all be expressed as partial derivatives of a single function lends an interest to the form and properties of this function. As a preliminary lemma in its investigation we remark that b_1, b_2, etc, k and θ may all be expressed as homogeneous functions of $c_1, c_2, c_3, c_4, c_5, c_6$. If we express the coordinates x', y', z', x, y, z, of the Sun and Moon by the general expression

$$x,\ y,\ z,\ x',\ y',\ z' = \Sigma\, k \begin{matrix}\sin\\\cos\end{matrix}(l + b\,t,) \tag{75}$$

the quantity k will necessarily be a homogeneous function of the first degree in a and a', the mean distances, and therefore in the language of physics, of dimensions L, while, since $b\,t$ is an angle, and so a pure number, b itself is of dimensions $\mathrm{T}^{-1} = \mathrm{M}^{\frac{1}{2}}\mathrm{L}^{-\frac{3}{2}}$ and therefore homogeneous and of the order $-\frac{3}{2}$ in a and a' when unit mass is defined as that which attracts with unit force at unit distance. Now, c_i is of the general form (62), that is, we have

$$\begin{aligned}
c_1 &= \frac{1}{2}\ \Sigma\ \mu\, i_1\, b\, k^2 \\
c_2 &= \frac{1}{2}\ \Sigma\ \mu\, i_2\, b\, k^2 \\
&\cdot \qquad \cdot \\
&\cdot \qquad \cdot \\
c_6 &= \frac{1}{2}\ \Sigma\ \mu\, i_6\, b\, k^2
\end{aligned} \tag{76}$$

$i_1, i_2, \ . \ . \ . \ i_6$ being the coefficients of $\lambda_1\ \lambda_2, \ . \ . \ . \ \lambda_6$ respectively in the angle $l + b\,t$, and μ being the functions of the masses employed in (62). Hence c_i considered as a function of

$$m_1,\ m_2,\ m_3,\ a,\ e,\ \gamma,\ a',\ e',\ \gamma',$$

will be homogeneous and of the degree $\frac{1}{2}$ in a and a'. Hence, conversely when we express these elements as functions of c_1, c_2, etc., these functions will be homogeneous and of the second degree in c_1, c_2, etc. Hence,

a, a', and k are functions of the degree 2 in c_1, c_2, etc.,
$b_1, b_2, \ . \ . \ b_6$ " " " -3 " "

and, since

$$d\,\Theta = b_1\, d\,c_1 + b_2\, d\,c_2 + \ . \ . \ . \ . \ + b_6\, d\,c_6$$

it follows that Θ is homogeneous and of the order -2 in c_1, c_2, etc.

The same reasoning may be expressed in another way. Let us suppose the linear dimensions of the system increased in the ratio $\nu : 1$, the masses, inclinations, and eccentricities all remaining unaltered. Then the analytical forms of the integrals of motion will remain unaltered, while the following changes will result in the values of the principal constants:—

All the values of k will be increased in the ratio $\nu : 1$
. . . . . . b . . . . $\nu^{-\frac{3}{2}} : 1$
. . . . . . c . . . . $\nu^{\frac{1}{2}} : 1$

These conditions can be fulfilled without changing the form of b and k only when b is homogeneous and of the order -3 and k homogeneous and of the order 2 in c. It follows that each value of k, b, and Θ must satisfy the partial differential equations

$$c_1\frac{d\,k}{d\,c_1}+c_2\frac{d\,k}{d\,c_2}+c_3\frac{d\,k}{d\,c_3}+c_4\frac{d\,k}{d\,c_4}+c_5\frac{d\,k}{d\,c_5}+c_6\frac{d\,k}{d\,c_6}-2\,k=0$$

$$c_1\frac{d\,b_i}{d\,c_1}+c_2\frac{d\,b_i}{d\,c_2}+c_3\frac{d\,b_i}{d\,c_3}+c_4\frac{d\,b_i}{d\,c_4}+c_5\frac{d\,b_i}{d\,b_5}+c_6\frac{d\,b_i}{d\,c_6}+3\,b_i=0$$

$$c_1\frac{d\,\Theta}{d\,c_1}+c_2\frac{d\,\Theta}{d\,c_2}+c_3\frac{d\,\Theta}{d\,c_3}+c_4\frac{d\,\Theta}{d\,c_4}+c_5\frac{d\,\Theta}{d\,c_5}+c_6\frac{d\,\Theta}{d\,c_6}+2\,\Theta=0$$

Since

$$b_1=\frac{d\,\Theta}{d\,c_1}\qquad b_2=\frac{d\,\Theta}{d\,c_2}\ \text{etc.}$$

we have, from the last equation

$$\Theta=-\frac{1}{2}\,(b_1\,c_1+b_2\,c_2+b_3\,c_3+b_4\,c_4+b_5\,c_5+b_6\,c_6)$$

If we substitute for c_1, c_2, etc., their values (76) this equation becomes

$$\Theta=-\frac{1}{4}\Sigma\,\mu\,(i_1\,b_1+i_2\,b_2+i_3\,b_3+i_4\,b_4+i_5\,b_5+i_6\,b_6)\,b\,k^2$$

or since

$$b=i_1\,b_1+i_2\,b_2+\text{etc.}$$

$$\Theta=-\frac{1}{4}\Sigma\,\mu\,b^2\,k^2$$

From this formula it can be shown that Θ is the negative of the function which Clausius has named the *viriel* of the system, that is, the constant term in the expression for the living force of the system of three bodies. Let us represent this living force by T. Then

$$2\,T=m_1\,(\xi'^2+\eta'^2_1+\zeta'^2_1)+m_2\,(\xi'^2_2+\eta'^2_2+\zeta'^2_2)+m_3\,(\xi'^2_3+\eta'^2_3+\zeta'^2_3)$$

the accents indicating derivatives as to t. If we differentiate the first three equations (5) with respect to the time, with ξ_0 a constant, and substitute for ξ, η, ζ their values in x, y, etc., we have

$$2\,T=\mu_1\left(\frac{d\,x'^2}{d\,t^2}+\frac{d\,y'^2}{d\,t^2}+\frac{d\,z'^2}{d\,t^2}\right)+\mu_2\left(\frac{d\,x^2}{d\,t^2}+\frac{d\,y^2}{d\,t^2}+\frac{d\,z^2}{d\,t^2}\right)$$

Differentiating the expression (75) we have

$$\frac{d}{d\,t}\left(x,\ y,\ z,\ x',\ y',\ \text{or}\ z'\right)=\Sigma\pm b\,k\,{\cos \atop \sin}\,(f+b\,t)$$

If we square each equation of this form, each term under the sign Σ will give in the square the constant term

$$\frac{1}{2}\,b^2\,k^2$$

Hence the sum of all the constants in 2 T, which we shall represent by 2 T_0, will be

$$\frac{1}{2} \Sigma \mu b^2 k^2$$

the sign Σ being extended so as to include all the terms in the six coordinates x', y', z', x, y, z. Comparing this with the above value of Θ we have

$$\Theta = -T_0$$

T_0 being the viriel of CLAUSIUS.

This result may be expressed as follows: Taking for the elements c_1, c_2, etc., the functions k_ϵ, k_π, k_θ, etc., and expressing the constant of the living force of the system composed of the Sun, Earth, and Moon as a function of k_ϵ, k_π, etc., the mean motions of the Sun and Moon, and the secular motions of their nodes and apsides will be given by the equations

$$\begin{gathered}
\frac{d\epsilon}{dt} + \frac{dT_0}{dk_\epsilon} = 0 \\
\frac{d\pi}{dt} + \frac{dT_0}{dk_\pi} = 0 \\
\frac{d\theta}{dt} + \frac{dT_0}{dk_\theta} = 0 \\
\frac{d\epsilon'}{dt} + \frac{dT_0}{dk'_\epsilon} = 0 \\
\frac{d\pi'}{dt} + \frac{dT_0}{dk'_\pi} = 0 \\
\frac{d\theta'}{dt} + \frac{dT_0}{dk'_\theta} = 0
\end{gathered} \tag{77}$$

These equations can not be directly used to obtain the values of these secular motions of ϵ, π, etc., because we can not form T_0 until we have solved the equations of motion. But, the solutions of these equations being once obtained, they may form a valuable check on their accuracy.

CHAPTER II.

SPECIAL CASE OF THE MOON'S MOTION.

§ 11.

COMPUTATION OF THE ANALYTIC EXPRESSIONS k_ϵ, k_π, ETC., EMPLOYED IN THE PRECEDING THEORY.

In the preceding chapters we have shown, in a general way, how we may deduce expressions for the simultaneous variations of the elements of the orbits of the Earth and of the Moon which are due to the action of a planet.

We now proceed to the actual calculation of these expressions. The coordinates of the Moon relatively to the Earth, and of the Sun relatively to the center of gravity of the Earth and Moon, are supposed to be given in advance, in terms of the time and of arbitrary constants. As already indicated, we shall take these data from the theory of DELAUNAY. In this work, however, the parallax is carried only to terms of the fifth order, while, for the present purpose, it is desired to carry the approximation to terms of a higher order.

During my visit to Paris in April, 1871, Mr. DELAUNAY was so good as to calculate the constant term of the parallax to terms of the sixth order, and afterward Professor J. C. ADAMS kindly communicated his results for this same term to the seventh and higher orders. I am thus enabled to present the actual analytic expressions for k_ϵ, k_π, etc., to terms of the order m^7 and $e'^2 m^5$, and to terms of the sixth order in e, γ, e' and m.

The notation of the constants is as follows:

The elements of the orbit of the moon, a, n, e, γ, have the same signification as in the theory of DELAUNAY, n being the actual mean motion of the Moon, and a the mean distance, derived from n by the equation

$$a^3 n^2 = m_2 + m_3$$

e is the eccentricity of the lunar orbit, and γ the sine of half inclination to the ecliptic. The coordinates of the moon relatively to the ecliptic are expressed in terms of g, g', λ, and λ', which represent respectively the mean anomalies of the Sun and Moon, and the mean angular distances of these bodies from the node of the Moon's orbit on the ecliptic.

These quantities are used for convenience instead of the l, l', F, and D of DELAUNAY, with which they are connected by the relations

$$\begin{aligned} l &= g \\ l' &= g' \\ F &= \lambda \\ D &= \lambda - \lambda' \end{aligned}$$

The required formulæ are presented in the form I have found most convenient for manipulation in the following pages Instead of writing in full each expression such as

$$C \begin{smallmatrix}\cos \\ \sin\end{smallmatrix} \left(i g + i' g' + j \lambda + j' \lambda' \right)$$

I use the plan of the German astronomers, simply writing the indices i, i', j, j', opposite the coefficient C of the term.

We commence with the value of sin P or $\frac{a}{r}$ given by DELAUNAY. If we put

$$\frac{a}{r} = 1 + \delta$$

we have

$$\frac{r}{a} = 1 - \delta + \delta^2 - \delta^3 + \text{etc.} \quad . \quad . \quad .$$

The several terms of $\frac{a}{r}$, so far as is necessary for our present purpose, are given in the first of the following series of tables. Only those terms are given which are necessary for determining k_e, k_π, and k_θ as high as m^7 and $e'^2 m^5$, and as high as the sixth dimension in all the other small quantities, as well as their combinations. Now, the mode of forming the quantities sought is such that it is not necessary to carry any coefficient to a greater degree of approximation than is necessary for obtaining its square to the necessary order of accuracy, with the single exception of the coefficient of $\cos g'$ in $\frac{a}{r}$. All the coefficients exceeding the third order are therefore omitted, and those retained are only given to such a degree of approximation that the sum of the lowest and highest orders of the terms of each separate coefficient shall be 6 or 7, according to circumstances. This remark applies to all the following expressions for the Moon's coordinates.

The several powers of $\frac{a}{r} - 1$ to the sixth, inclusive, being formed, they are combined according to the preceding formulæ to obtain the value of $\frac{r}{a}$, the coefficients for which are given in the next table.

The value of β, or the Moon's latitude, as given by DELAUNAY, follows. The several powers of β being formed, sin β and cos β are obtained by the formulæ

$$\sin \beta = \beta - \frac{1}{6} \beta^3 + \frac{1}{120} \beta^5,$$

$$\cos \beta = 1 - \frac{1}{2} \beta^2 + \frac{1}{24} \beta^4 - \frac{1}{720} \beta^6,$$

for which the expressions are given.

These quantities are then multiplied by $\frac{r}{a}$, and the products are given in the next two tables. The expressions for $\cos \delta v$ and $\sin \delta v$ are formed in the same way from δv, and their products by $\frac{r}{a} \cos \beta$ are given. We thus have the coefficients of $\cos l$ and $\sin l$ in the expressions (48) for ξ and η, the rectangular coordinates of the Moon referred to the ecliptic.

cos				
g,	g',	λ,	λ'	$\frac{a}{r} = 1 +$
0	0	0	0	$\frac{1}{6} m^2 + \frac{1}{4} e'^2 m^2 - \frac{179}{288} m^4 - \frac{97}{48} m^5 + \frac{1}{16} e^4 m^2 + \frac{5}{16} e'^4 m^2 + 2 e^2 \gamma^2 m^2 - \gamma^4 m^2 + \frac{3}{16} \alpha^2 m^2 - \frac{799}{192} e'^2 m^4 - \frac{757}{162} m^6 - \frac{873}{32} e'^2 m^5 - \frac{4039}{432} m^7$
1	0	0	0	$e - \frac{1}{8} e^3 - \frac{7}{12} e m^2 - \frac{285}{64} e m^3 + \frac{5}{2} e \gamma^4 - \frac{5}{4} e^3 \gamma^2 + \frac{1}{192} e^5 + \frac{19}{32} e \gamma^2 m^2 + \frac{19}{96} e^3 m^2 - \frac{7}{8} e e'^2 m^2 - \frac{45091}{2304} e m^4$
2	0	0	0	$e^2 - \frac{1}{3} e^4 - \frac{5}{6} e^2 m^2$
3	0	0	0	$\frac{9}{8} e^3$
−1	1	0	0	$\frac{21}{8} e e' m$
0	1	0	0	$-\frac{3}{2} e' m^2 + \frac{449}{16} e' m^4$
1	1	0	0	$-\frac{21}{8} e e' m$
−1	0	2	0	$-\frac{5}{2} e \gamma^2$
0	0	2	0	$-5 e^2 \gamma^2$
−2	0	2	−2	$-\frac{15}{14} e^2 m^2$
−1	0	2	−2	$\frac{15}{8} e m + \frac{187}{32} e m^2 - \frac{15}{4} e \gamma^2 m - \frac{75}{16} e e'^2 m + \frac{29513}{1536} e m^3$
0	0	2	−2	$m^2 + \frac{15}{4} e^2 m + \frac{19}{6} m^3 + \frac{189}{16} e^2 m^2 - \frac{5}{2} e'^2 m^2 - 2 \gamma^2 m^2 + \frac{131}{18} m^4 - \frac{239}{12} e'^2 m^3 + \frac{383}{27} m^5$

cos $g,\ g',\ \lambda,\ \lambda'$	$\frac{a}{r} = 1 +$
1 0 2 −2	$\frac{33}{16} e m^2$
−1 −1 2 −2	$\frac{35}{8} e e' m$
0 −1 2 −2	$\frac{7}{2} e' m^2 + \frac{157}{8} e' m^3$
−1 1 2 −2	$-\frac{15}{8} e e' m$
0 1 2 −2	$-\frac{1}{2} e' m^2 - \frac{91}{24} e' m^3$
0 1 1 −1	$\frac{5}{4} e' \alpha$
0 0 1 −1	$-\frac{15}{16} \alpha m$

From this we obtain

cos $g,\ g',\ \lambda,\ \lambda'$	$\frac{r}{a} = 1 +$
0 0 0 0	$\frac{1}{2} e^2 - \frac{1}{6} m^2 + \frac{355}{384} e^2 m^2 - \frac{1}{4} e'^2 m^2 + \frac{331}{288} m^4 + \frac{1905}{256} e^2 m^3 + \frac{83}{16} m^5 - \frac{5}{4} e^4 \gamma^2$ $+ \frac{45}{8} e^2 \gamma^4 + \frac{25}{32} e'^2 \alpha^2 + \frac{225}{128} e^4 m^2 - \frac{5}{16} e'^4 m^2 + \gamma^4 m^2$ $+ \frac{1047}{128} e^2 e'^2 m^2 - \frac{135}{16} e^2 \gamma^2 m^2 + \frac{129}{512} \alpha^2 m^2 + \frac{1\,499\,551}{36\,864} e^2 m^4$ $+ \frac{1751}{192} e'^2 m^4 - 2 \gamma^2 m^4 + \frac{42\,775}{2592} m^6 + \frac{2241}{32} e'^2 m^5 + \frac{4787}{108} m^7$
1 0 0 0	$-e + \frac{3}{8} e^3 + \frac{11}{12} e m^2 + \frac{405}{64} e m^3 - \frac{5}{192} e^5 + \frac{5}{2} e \gamma^4 + \frac{5}{4} e^3 \gamma^2 + \frac{181}{128} e^3 m^2$ $+ \frac{11}{8} e e'^2 m^2 - \frac{19}{32} e \gamma^2 m^2 + \frac{70\,027}{2304} e m^4$
2 0 0 0	$-\frac{1}{2} e^2 + \frac{1}{3} e^4 + \frac{1}{3} e^2 m^2$
3 0 0 0	$-\frac{3}{8} e^3$

cos				$\frac{r}{a}$ (continued).
g,	g',	λ,	λ'	
-1	1	0	0	$-\frac{21}{8}ee'm$
0	1	0	0	$\frac{3}{2}e'm^2-\frac{409}{16}e'm^4$
1	1	0	0	$\frac{21}{8}ee'm$
-1	0	2	0	$\frac{5}{2}e\gamma^2$
0	0	2	0	$\frac{5}{2}e^2\gamma^2$
-2	0	2	-2	$\frac{15}{8}e^2m$
-1	0	2	-2	$-\frac{15}{8}em-\frac{155}{32}em^2+\frac{15}{16}e^3m+\frac{75}{16}ee'^2m+\frac{15}{4}e\gamma^2m-\frac{23\,689}{1536}em^3$
0	0	2	-2	$-m^2-\frac{15}{8}e^2m-\frac{19}{6}m^3-\frac{173}{32}e^2m^2+\frac{5}{2}e'^2m^2+2\gamma^2m^2-\frac{125}{18}m^4$ $+\frac{239}{12}e'^2m^3-\frac{709}{54}m^5$
1	0	2	-2	$-\frac{17}{16}em^2$
-1	-1	2	-2	$-\frac{35}{8}ee'm$
0	-1	2	-2	$-\frac{7}{2}e'm^2-\frac{157}{8}e'm^3$
-1	1	2	-2	$\frac{15}{8}ee'm$
0	1	2	-2	$\frac{1}{2}e'm^2+\frac{91}{24}e'm^3$
0	0	1	-1	$\frac{15}{16}\alpha m$
0	1	1	-1	$-\frac{5}{4}e'\alpha$

We have from Delaunay

sin g,	g',	λ,	λ'	$\beta=\gamma\times$
-2	0	1	0	$-\frac{3}{2}e^2$
-1	0	1	0	$-2e+\frac{5}{4}e^3-5e\gamma^2+\frac{189}{32}em^2$
0	0	1	0	$2-2e^2+\frac{7}{32}e^4-\frac{1}{4}\gamma^4$
1	0	1	0	$2e-\frac{5}{2}e^3-\frac{1}{2}em^2$
2	0	1	0	$\frac{9}{4}e^2$
0	-1	1	0	$\frac{3}{4}e'm$
0	1	1	0	$-\frac{3}{4}e'm$
0	0	3	0	$-\frac{1}{3}\gamma^2$
-1	0	3	-2	$\frac{15}{4}em$
0	0	3	-2	$\frac{11}{8}m^2$
0	-1	1	-2	$\frac{7}{4}e'm$
-1	0	1	-2	$3em$
0	0	1	-2	$\frac{3}{4}m+\frac{25}{16}m^2+\frac{27}{16}e^2m-\frac{15}{8}e'^2m+\frac{9}{8}\gamma^2m+\frac{2917}{768}m^3$
1	0	1	-2	$\frac{3}{4}em$
0	1	1	-2	$-\frac{3}{4}e'm$

From this we obtain

sin $g,\ g',\ \lambda,\ \lambda'$	$\sin\beta=$
$-2\ \ 0\ \ 1\ \ 0$	$-\frac{3}{2}e^2\gamma$
$-1\ \ 0\ \ 1\ \ 0$	$-2e\gamma-4e\gamma^3+\frac{5}{4}e^3\gamma+\frac{189}{32}e\gamma m^2$
$0\ \ 0\ \ 1\ \ 0$	$2\gamma-2e^2\gamma-\gamma^3+\frac{7}{32}e^4\gamma+e^2\gamma^3-\frac{1}{4}\gamma^5-\frac{9}{32}\gamma^3m^2$
$1\ \ 0\ \ 1\ \ 0$	$2e\gamma-\frac{5}{2}e^3\gamma-e\gamma^3-\frac{1}{2}e\gamma m^2$
$2\ \ 0\ \ 1\ \ 0$	$\frac{9}{4}e^2\gamma$
$0\ -1\ \ 1\ \ 0$	$\frac{3}{4}e'\gamma m$
$0\ \ 1\ \ 1\ \ 0$	$-\frac{3}{4}e'\gamma m$
$-1\ \ 0\ \ 3\ -2$	$\frac{15}{4}e\gamma m$
$0\ \ 0\ \ 3\ -2$	$\frac{11}{8}\gamma m^2$
$0\ -1\ \ 1\ -2$	$\frac{7}{4}e'\gamma m$
$-1\ \ 0\ \ 1\ -2$	$3e\gamma m$
$0\ \ 0\ \ 1\ -2$	$\frac{3}{4}\gamma m+\frac{25}{16}\gamma m^2+\frac{27}{16}e^2\gamma m-\frac{15}{8}e'^2\gamma m+\frac{3}{8}\gamma^3m+\frac{2957}{768}\gamma m^3$
$1\ \ 0\ \ 1\ -2$	$\frac{3}{4}e\gamma m$
$0\ \ 1\ \ 1\ -2$	$-\frac{3}{4}e'\gamma m$

cos $g,\ g',\ \lambda,\ \lambda'$	$\cos\beta=1+$
$0\ \ 0\ \ 0\ \ 0$	$-\gamma^2+\frac{1}{4}\gamma^4-\frac{9}{64}\gamma^2m^2-\frac{75}{128}\gamma^2m^3+\frac{45}{64}e\ \ \gamma^2-5e^2\gamma^4+\frac{1}{4}\gamma^6$ $-\frac{17}{128}e^2\gamma^2m^2-\frac{31}{64}e'^2\gamma^2m^2-\frac{9}{32}\gamma^4m^2-\frac{5175}{2048}\gamma^2m^4$
$1\ \ 0\ \ 0\ \ 0$	$-\frac{5}{2}e^3\gamma^2+5e\gamma^4-\frac{109}{16}e\gamma^2m^2$

cos				$\cos\beta$ (continued).
g,	g',	λ,	λ'	
2	0	0	0	$\frac{5}{4}e^2\gamma^2$
0	1	0	0	0
-2	0	2	0	$-\frac{1}{2}e^2\gamma^2$
-1	0	2	0	$-2e\gamma^2$
0	0	2	0	$\gamma^2-4e^2\gamma^2$
1	0	2	0	$2e\gamma^2$
2	0	2	0	$\frac{13}{4}e^2\gamma^2$
-1	0	2	-2	$-\frac{3}{2}e\gamma^2 m$
0	0	2	-2	$\frac{3}{4}\gamma^2 m+\frac{3}{16}\gamma^2 m^2$
1	0	2	-2	$\frac{3}{2}e\gamma^2 m$

sin				$\frac{r}{a}\sin\beta=$
g,	g',	λ,	λ'	
-2	0	1	0	$-e^2\gamma$
-1	0	1	0	$-3e\gamma-6e\gamma^3+\frac{15}{8}e^3\gamma+\frac{503}{64}e\gamma m^2$
0	0	1	0	$2\gamma-e^2\gamma-\gamma^3-\frac{1}{3}\gamma m^2+\frac{3}{8}\gamma m^3-\frac{11}{32}e^4\gamma+3e^2\gamma^3-\frac{1}{4}\gamma^5-\frac{25}{48}e^2\gamma m^2$ $-\frac{1}{2}e'^2\gamma m^2-\frac{11}{96}\gamma^3 m^2+\frac{1031}{288}\gamma m^4$
1	0	1	0	$e\gamma-\frac{3}{4}e^3\gamma-\frac{1}{2}e\gamma^3+\frac{1}{12}e\gamma m^2$
2	0	1	0	$\frac{3}{4}e^2\gamma$
0	-1	1	0	$\frac{3}{4}e'\gamma m$

sin				
g	g'	λ	λ'	$\frac{r}{a}\sin\beta$ (continued).
0	1	1	0	$-\frac{3}{4}e'\gamma m$
−1	0	3	−2	$\frac{15}{8}e\gamma m$
0	0	3	−2	$\frac{3}{8}\gamma m^2$
0	−1	1	−2	$\frac{7}{4}e'\gamma m$
−1	0	1	−2	$\frac{9}{2}e\gamma m$
0	0	1	−2	$\frac{3}{4}\gamma m+\frac{41}{16}\gamma m^2+\frac{3}{8}\gamma^3 m+\frac{3}{16}e^2\gamma m-\frac{15}{8}e'^2\gamma m+\frac{5293}{768}\gamma m^3$
1	0	1	−2	$\frac{3}{8}e\gamma m$
0	1	1	−2	$-\frac{3}{4}e'\gamma m$

cos				
g	g'	λ	λ'	$\frac{r}{a}\cos\beta = 1+$
0	0	0	0	$\frac{1}{2}e^2-\gamma^2-\frac{1}{6}m^2-\frac{1}{2}e^2\gamma^2+\frac{1}{4}\gamma^4+\frac{355}{384}e^2m^2+\frac{5}{192}\gamma^2m^2-\frac{1}{4}e'^2m^2$ $+\frac{331}{288}m^4+\frac{1905}{256}e^2m^3-\frac{123}{128}\gamma^2m^3+\frac{83}{16}m^5+\frac{25}{64}e^4\gamma^2$ $-3e^2\gamma^4+\frac{1}{4}\gamma^6-\frac{2095}{384}e^2\gamma^2m^2+\frac{1047}{128}e^2e'^2m^2-\frac{15}{64}e'^2\gamma^2m^2$ $+\frac{225}{18}e^4m^2+\frac{65}{96}\gamma^4m^2-\frac{5}{16}e'^4m^2+\frac{1\,499\,551}{36\,864}e^2m^4$ $-\frac{127\,807}{18\,432}\gamma^2m^4+\frac{1751}{192}e'^2m^4+\frac{42\,775}{2592}m^6+\frac{2241}{32}e'^2m^5$ $+\frac{4787}{108}m^7+\frac{25}{32}e'^2\alpha^2+\frac{129}{512}\alpha^2m^2$
1	0	0	0	$-e+\frac{3}{8}e^3+e\gamma^2+\frac{11}{12}em^2+\frac{405}{64}em^3-\frac{5}{192}e^5-\frac{9}{4}e^3\gamma^2+\frac{7}{2}e\gamma^4$ $+\frac{181}{128}e^3m^2-\frac{853}{96}e\gamma^2m^2+\frac{11}{8}ee'^2m^2+\frac{70\,027}{2304}em^4$

cos				$\frac{r}{a}\cos\beta = 1 +$
g,	g',	λ,	λ'	
2	0	0	0	$-\frac{1}{2}e^2+\frac{1}{3}e^4+\frac{7}{4}e^2\gamma^2+\frac{1}{3}e^2m^2$
3	0	0	0	$-\frac{3}{8}e^3$
−1	1	0	0	$-\frac{21}{8}ee'm$
0	1	0	0	$\frac{3}{2}e'm^2-\frac{409}{16}e'm^4$
1	1	0	0	$\frac{21}{8}ee'm$
−1	0	2	0	0
0	0	2	0	$\gamma^2-e^2\gamma^2-\frac{1}{6}\gamma^2m^2$
1	0	2	0	$\frac{3}{2}e\gamma^2$
−2	0	2	−2	$\frac{15}{8}e^2m$
−1	0	2	−2	$-\frac{15}{8}em-\frac{155}{32}em^2$
0	0	2	−2	$-m^2-\frac{15}{8}e^2m+\frac{3}{4}\gamma^2m-\frac{19}{6}m^3-\frac{173}{32}e^2m^2+\frac{51}{16}\gamma^2m^2+\frac{5}{2}e'^2m^2$ $-\frac{125}{18}m^4+\frac{239}{12}e'^2m^3-\frac{709}{54}m^5$
1	0	2	−2	$-\frac{17}{16}em^2$
−1	−1	2	−2	$-\frac{35}{8}ee'm$
0	−1	2	−2	$-\frac{7}{2}e'm^2-\frac{157}{8}e'm^3$
−1	1	2	−2	$\frac{15}{8}ee'm$
0	1	2	−2	$\frac{1}{2}e'm^2+\frac{91}{24}e'm^3$

cos				
g	g'	λ	λ'	$\frac{r}{a}\cos\beta$ (continued).
0	0	1	-1	$\frac{15}{16}\alpha m$
0	1	1	-1	$-\frac{5}{4}e'\alpha$
0	0	0	2	$-\frac{3}{4}\gamma^2 m$

We have also from DELAUNAY

sin				
g	g'	λ	λ'	$\delta v =$
1	0	0	0	$2e-\frac{1}{4}e^3+\frac{5}{96}e^5$
2	0	0	0	$\frac{5}{4}e^2-\frac{11}{24}e^4-\frac{5}{4}e^2\gamma^2-\frac{7}{16}e^2m^2$
3	0	0	0	$\frac{13}{12}e^3$
-1	1	0	0	$-\frac{21}{4}ee'm$
0	1	0	0	$-3e'm-\frac{27}{8}e^2e'm+\frac{27}{2}\gamma^2e'm-\frac{27}{8}e'^3m+\frac{735}{16}e'm^3+\frac{1261}{4}e'm^4$
1	1	0	0	$-\frac{21}{4}ee'm$
0	2	0	0	$-\frac{9}{4}e'^2m$
-1	0	2	0	$-3e\gamma^2$
0	0	2	0	$\gamma^2-\frac{9}{4}e^2\gamma^2-\gamma^4+\frac{11}{4}\gamma^2m^2$
1	0	2	0	$-2e\gamma^2$
-2	0	2	-2	$\frac{45}{16}e^2m$
-1	0	2	-2	$\frac{15}{4}em+\frac{263}{16}em^2-6e\gamma^2m-\frac{75}{8}ee'^2m+\frac{48\,217}{768}em^3$

sin				δv (continued).
$g,$	$g',$	$\lambda,$	λ'	
0	0	2	−2	$\frac{11}{8}m^2+\frac{75}{16}e^2m-\frac{3}{4}\gamma^2m+\frac{59}{12}m^3+\frac{1101}{64}e^2m^3-\frac{47}{16}\gamma^2m^2-\frac{55}{16}e'^2m^2+\frac{893}{72}m^4-\frac{691}{24}e'^2m^3+\frac{2855}{108}m^5$
1	0	2	−2	$\frac{17}{8}em^2$
−1	−1	2	−2	$\frac{35}{4}ee'm$
0	−1	2	−2	$\frac{77}{16}e'm^2+\frac{479}{16}e'm^3$
−1	1	2	−2	$-\frac{15}{4}ee'm$
0	1	2	−2	$-\frac{11}{16}e'm^2-\frac{257}{48}e'm^3$
0	0	0	2	$-\frac{9}{4}\gamma^2m$
0	1	1	−1	$\frac{5}{2}e'\alpha$
0	0	1	−1	$-\frac{15}{8}\alpha m$

From this we derive

cos				$\cos\delta v=1+$
$g,$	$g,$	$\lambda,$	λ'	
0	0	0	0	$-e^2+\frac{7}{64}e^4-\frac{1}{4}\gamma^4-\frac{225}{64}e^2m^2-\frac{9}{4}e'^2m^2-\frac{121}{256}m^4-\frac{8715}{256}e^2m^3+\frac{33}{64}\gamma^2m-\frac{649}{192}m^5-\frac{5}{288}e^6+\frac{25}{32}e^4\gamma^2-\frac{33}{8}e^2\gamma^4-\frac{1}{2}\gamma^6-\frac{1885}{512}e^4m^2-\frac{1387}{64}e^3e'^2m^2-\frac{405}{64}e'^4m^2+\frac{1665}{128}e^2\gamma^2m^2+\frac{81}{4}e'^2\gamma^2m^2-\frac{1}{32}\gamma^4m^2-\frac{428\,589}{2048}e^2m^4+\frac{33\,465}{512}e'^2m^4+\frac{989}{256}\gamma^2m^4-\frac{1865}{128}m^6-\frac{25}{16}e'^2\alpha^2-\frac{225}{256}\alpha^2m^2+\frac{54\,687}{128}e'^2m^5-\frac{21\,023}{432}m^7$

cos g,	 g,	 λ,	 λ'	cos δv (continued).
1	0	0	0	$-\frac{5}{4}e^3-\frac{165}{64}em^3+\frac{17}{48}e^5+\frac{5}{4}e^3\gamma^2-\frac{5}{2}e\gamma^4-\frac{109}{8}e^3m^2-\frac{63}{4}ee'^2m^2$ $+\frac{45}{32}e\gamma^2m^2-\frac{5627}{256}em^4$
2	0	0	0	$e^2-\frac{5}{3}e^4$
3	0	0	0	$\frac{5}{4}e^3$
−1	1	0	0	$3ee'm$
0	1	0	0	$-\frac{363}{128}e'm^4$
1	−1	0	0	$-3ee'm$
−1	0	2	0	$e\gamma^2$
0	0	2	0	$-e^2\gamma^2$
1	0	2	0	$-e\gamma^2$
−2	0	2	−2	$-\frac{15}{4}e^2m$
−1	0	2	−2	$-\frac{11}{8}em^2-\frac{15}{8}e^3m-\frac{59}{12}em^3+\frac{3}{4}e\gamma^2m$
0	0	2	−2	$\frac{15}{4}e^2m+\frac{229}{16}e^2m^2-\frac{33}{4}e'^2m^3$
1	0	2	−2	$\frac{11}{8}em^2$
0	−1	2	−2	$\frac{35}{4}e^2e'm+\frac{33}{16}e'm^3$
0	1	2	−2	$-\frac{15}{4}e^2e'm-\frac{33}{16}e'm^3$

sin g,	 g',	 λ,	 λ',	sin $\delta v=$
1	0	0	0	$2e-\frac{5}{4}e^3+\frac{17}{48}e^5-\frac{1}{2}e\gamma^4-\frac{225}{32}e^3m^2-\frac{9}{2}ee'^2m^2-\frac{121}{128}em^4$

sin				$\sin \delta v =$
g,	g',	λ,	λ'	
2	0	0	0	$\frac{5}{4}e^2-\frac{41}{24}e^4-\frac{5}{4}e^2\gamma^2-\frac{7}{16}e^2m^2$
3	0	0	0	$\frac{17}{12}e^3$
−1	1	0	0	$-\frac{21}{4}ee'm$
0	1	0	0	$-3e'm-\frac{3}{8}e^2e'm+\frac{27}{2}\gamma^2e'm-\frac{27}{8}e'^3m-\frac{735}{16}e'm^3+\frac{1261}{4}e'm^4$
1	1	0	0	$-\frac{21}{4}ee'm$
0	2	0	0	$-\frac{9}{4}e'^2m$
−1	0	2	0	$-3e\gamma^2$
0	0	2	0	$-\gamma^2-\frac{5}{4}e^2\gamma^2-\gamma^4+\frac{11}{4}\gamma^2m^2$
1	0	2	0	$-2e\gamma^2$
−2	0	2	−2	$\frac{45}{16}e^2m$
−1	0	2	−2	$\frac{15}{4}em+\frac{263}{16}em^2-\frac{15}{4}e^3m-6e\gamma^2m-\frac{75}{8}ee'^2m+\frac{48217}{768}em^3$
0	0	2	−2	$\frac{11}{8}m^2+\frac{75}{16}e^2m-\frac{3}{4}\gamma^2m+\frac{59}{12}m^3+\frac{1013}{64}e^2m^2-\frac{47}{16}\gamma^2m^2-\frac{55}{16}e'^2m^2$ $+\frac{893}{72}m^4-\frac{691}{24}e'^2m^3+\frac{2855}{108}m^5$
1	0	2	−2	$\frac{17}{8}em^2$
−1	−1	2	−2	$\frac{35}{4}ee'm$
0	−1	2	−2	$\frac{77}{16}e'm^2+\frac{479}{16}e'm^3$
−1	1	2	−2	$-\frac{15}{4}ee'm$

sin				
g,	g',	λ,	λ'	$\sin \delta v =$
0	1	2	−2	$-\frac{11}{16} e' m^2 - \frac{257}{48} e' m^3$
0	0	0	2	$-\frac{9}{4} \gamma^2 m$
0	1	1	−1	$\frac{5}{2} e' \alpha$
0	0	1	−1	$-\frac{15}{8} \alpha m$

Multiplying by $\frac{r}{a} \cos \beta$ we find

cos				
g,	g',	λ,	λ'	$\frac{r}{a} \cos \beta \cos \delta v = 1 +$
0	0	0	0	$-\frac{1}{2} e^2 - \gamma^2 - \frac{1}{6} m^2 - \frac{1}{64} e^4 + \frac{1}{2} e^2 \gamma^2 - \frac{931}{384} e^2 m^2 + \frac{5}{192} \gamma^2 m^2 - \frac{5}{2} e'^2 m^2$ $+ \frac{1559}{2304} m^4 - \frac{3315}{128} e^2 m^3 - \frac{57}{128} \gamma^2 m^3 + \frac{347}{192} m^5 - \frac{29}{1152} e^6$ $+ \frac{19}{16} e^4 \gamma^2 - \frac{15}{2} e^2 \gamma^4 - \frac{5363}{1536} e^4 m^2 + \frac{265}{24} e^2 \gamma^2 m^2 - \frac{425}{64} e'^4 m^2$ $- \frac{1839}{128} e^2 e'^2 m^2 + \frac{11}{16} \gamma^2 m^2 + \frac{1425}{64} e'^2 \gamma^2 m^2 - \frac{6\,056\,323}{36\,864} e^2 m^4$ $- \frac{47\,887}{18\,432} \gamma^2 m^4 + \frac{114\,979}{1536} e'^2 m^4 + \frac{83\,407}{41\,472} m^6 - \frac{25}{32} e'^2 \alpha^2 - \frac{321}{512} \alpha^2 m^2$ $+ \frac{63\,651}{128} e'^2 m^5 - \frac{4351}{1152} m^7$
1	0	0	0	$-e - \frac{3}{8} e^3 + e \gamma^2 + \frac{11}{12} e m^2 + \frac{15}{4} e m^3 + \frac{5}{96} e^5 - \frac{1}{4} e^3 \gamma^2 + \frac{5}{4} e \gamma^4$ $- \frac{1145}{128} e^3 m^2 - \frac{359}{48} e \gamma^2 m^2 - \frac{97}{8} e e'^2 m^2 + \frac{20\,473}{2304} e m^4$
2	0	0	0	$\frac{1}{2} e^2 - \frac{1}{3} e^4 + \frac{3}{4} e^2 \gamma^2 + \frac{1}{6} e^2 m^2$
3	0	0	0	$\frac{3}{8} e^3$
−1	1	0	0	$\frac{3}{8} e e' m$
0	1	0	0	$\frac{3}{2} e' m^2 - \frac{3635}{128} e' m^4$

cos				$\frac{r}{a}\cos\beta\cos\delta v = 1 +$
g,	g',	λ,	λ'	
1	1	0	0	$-\frac{3}{8}ee'm$
−1	0	2	0	$e\gamma^2$
0	0	2	0	$\gamma^2-3e^2\gamma^2-\frac{1}{6}\gamma^2m^2$
1	0	2	0	$\frac{1}{2}e\gamma^2$
−2	0	2	−2	$-\frac{15}{8}e^2m$
−1	0	2	−2	$-\frac{15}{8}em-\frac{199}{32}em^2+\frac{9}{2}e\gamma^2m+\frac{75}{16}ee'^2m+\frac{15}{16}e^3m-\frac{31\,241}{1536}em^3$
0	0	2	−2	$-m^2+\frac{15}{8}e^2m+\frac{3}{4}\gamma^2m-\frac{19}{6}m^3+\frac{317}{32}e^2m^2+\frac{51}{16}\gamma^2m^2+\frac{5}{2}e'^2m^2$ $-\frac{125}{18}m^4+\frac{35}{3}e'^2m^3-\frac{709}{54}m^5$
1	0	2	−2	$\frac{5}{16}em^2$
−1	−1	2	−2	$-\frac{35}{8}ee'm$
0	−1	2	−2	$-\frac{7}{2}e'm^2-\frac{281}{16}e'm^3$
−1	1	2	−2	$\frac{15}{8}ee'm$
0	1	2	−2	$\frac{1}{2}e'm^2+\frac{83}{48}e'm^3$
0	0	0	2	$-\frac{3}{4}\gamma^2m$
0	0	1	−1	$\frac{15}{16}\alpha m$
0	1	1	−1	$-\frac{5}{4}e'\alpha$

sin				
g,	g',	λ,	λ'	$\frac{r}{a}\cos\beta\sin\delta v=$
1	0	0	0	$2e-\frac{3}{8}e^3-2e\gamma^2-\frac{1}{3}em^2+\frac{75}{128}em^3+\frac{5}{96}e^5-\frac{1}{4}e^3\gamma^2+\frac{5}{4}e\gamma^4$
				$+\frac{97}{128}e^3m^2-\frac{125}{192}e\gamma^2m^2+\frac{23}{8}ee'^2m^2+\frac{33\,353}{4608}em^4$
2	0	0	0	$\frac{1}{4}e^2-\frac{15}{12}e^4-\frac{3}{2}e^2\gamma^2+\frac{13}{48}e^2m^2$
3	0	0	0	$\frac{7}{24}e^3$
−1	1	0	0	$-\frac{15}{4}ee'm$
0	1	0	0	$-3e'm-\frac{15}{8}e^2e'm+\frac{33}{2}\gamma^2e'm-\frac{27}{8}e'^3m+\frac{743}{16}e'm^3+\frac{1261}{4}e'm^4$
1	1	0	0	$-\frac{15}{4}ee'm$
0	2	0	0	$-\frac{9}{4}e'^2m$
−1	0	2	0	$-\frac{7}{2}e\gamma^2$
0	0	2	0	$-\gamma^2-\frac{3}{4}e^2\gamma^2+\frac{35}{12}\gamma^2m^2$
1	0	2	0	$-\frac{1}{2}e\gamma^2$
−2	0	2	−2	$\frac{45}{16}e^2m$
−1	0	2	−2	$\frac{15}{4}em+\frac{67}{4}em^2-\frac{15}{8}e^3m-\frac{81}{8}e\gamma^2m-\frac{75}{8}ee'^2m+\frac{48\,281}{768}em^3$
0	0	2	−2	$\frac{11}{8}m^2+\frac{15}{16}e^2m-\frac{3}{4}\gamma^2m+\frac{59}{12}m^3+\frac{221}{64}e^2m^2-\frac{69}{16}\gamma^2m^2-\frac{55}{16}e'^2m^2$
				$+\frac{1753}{144}m^4-\frac{547}{24}e'^2m^3+\frac{5533}{216}m^5$
1	0	2	−2	$\frac{7}{16}em^2$
−1	−1	2	−2	$\frac{35}{4}ee'm$

sin g	g'	λ	λ'	$\frac{r}{a}\cos\beta\sin\delta v$ (continued).
0	−1	2	−2	$\frac{77}{16}e'm^2+\frac{455}{16}e'm^3$
−1	1	2	−2	$-\frac{15}{4}ee'm$
0	1	2	−2	$-\frac{11}{16}e'm^2-\frac{185}{48}e'm^3$
0	0	0	2	$-\frac{9}{4}\gamma^2 m$
0	1	1	−1	$\frac{5}{2}e'\alpha$
0	0	1	−1	$-\frac{15}{8}\alpha m$

The values of $k_{,}$, $k_{,,}$, etc., may be formed from these expressions without going through the process of forming the rectangular coordinates x, y, z, in full. Let us consider any term in the value of $\frac{r}{a}\cos\beta\cos.\delta v$, the corresponding term in the value of $\frac{r}{a}\cos\beta\sin\delta v$, and any term whatever in $\frac{r}{a}\sin\beta$. Let these terms be

$$\frac{r}{a}\cos\beta\cos\delta v = k\cos \mathrm{N}$$

$$\frac{r}{a}\cos\beta\sin\delta v = k'\sin \mathrm{N}$$

$$\frac{r}{a}\sin\beta \qquad = c\sin \mathrm{N}'$$

k and k' representing the several coefficients given on the preceding pages. If we substitute these values in the second members of (48) p. 129, we find

$$\frac{\xi}{a}=\frac{1}{2}(k+k')\cos(l+\mathrm{N})+\frac{1}{2}(k-k')\cos(l-\mathrm{N})$$

$$\frac{\eta}{a}=\frac{1}{2}(k+k')\sin(l+\mathrm{N})+\frac{1}{2}(k-k')\sin(l-\mathrm{N})$$

$$\frac{\zeta}{a}=c\sin \mathrm{N}'$$

The general expressions for the rectangular coordinates of the Moon, referred to any system of axes having their origin at the center of the Earth are

$$x=\mathrm{a}\;\xi+\mathrm{b}\;\eta+\mathrm{c}\;\zeta$$
$$y=\mathrm{a}'\;\xi+\mathrm{b}'\;\eta+\mathrm{c}'\;\zeta$$
$$z=\mathrm{a}''\;\xi+\mathrm{b}''\;\eta+\mathrm{c}''\;\zeta$$

the direction cosines a, b, . . . having the values given in (51) p. 131. If we put

$$\begin{aligned} k_1&=\frac{1}{2}(k+k') \\ k_2&=\frac{1}{2}(k-k') \end{aligned} \tag{78}$$

we find

$$\begin{aligned} \frac{x}{a}=&k_1(1-\gamma'^2)\cos(\theta'-\tau+l+\mathrm{N}) \\ &+k_2(1-\gamma'^2)\cos(\theta'-\tau+l-\mathrm{N}) \\ &+k_1\gamma'^2\cos(\theta'+\tau-l-\mathrm{N}) \\ &+k_2\gamma'^2\cos(\theta'+\tau-l+\mathrm{N}) \\ &-c\gamma'\sqrt{1-\gamma'^2}\cos(\theta'+\mathrm{N}') \\ &+c\gamma'\sqrt{1-\gamma'^2}\cos(\theta'-\mathrm{N}') \end{aligned} \tag{79}$$

$$\begin{aligned} \frac{y}{a}=&k_1(1-\gamma'^2)\sin(\theta'-\tau+l+\mathrm{N}) \\ &+k_2(1-\gamma'^2)\sin(\theta'-\tau+l-\mathrm{N}) \\ &+k_1\gamma'^2\sin(\theta'+\tau-l-\mathrm{N}) \\ &+k_2\gamma'^2\sin(\theta'+\tau-l+\mathrm{N}) \\ &-c\gamma'\sqrt{1-\gamma'^2}\sin(\theta'+\mathrm{N}') \\ &+c\gamma'\sqrt{1-\gamma'^2}\sin(\theta'-\mathrm{N}') \end{aligned}$$

$$\begin{aligned} \frac{z}{a}=&2\,k_1\gamma'\sqrt{1-\gamma'^2}\sin(l-\tau+\mathrm{N}) \\ &+2\,k_2\gamma'\sqrt{1-\gamma'^2}\sin(l-\tau-\mathrm{N}) \\ &+c(1-2\,\gamma'^2)\sin\mathrm{N}' \end{aligned}$$

Representing the coefficients of g, g', λ, λ' in N by i, i', j, j', respectively, we have

$$\begin{aligned} \mathrm{N}&=i\;g+i'\;g'+j\;\lambda+j'\;\lambda' \\ \mathrm{N}'&=i_1 g+i'_1 g'+j_1\lambda+j'_1\lambda' \end{aligned} \tag{80}$$

Substituting these values of N and N′, after effecting the substitutions (53) to reduce the expressions to the form (54), remembering that

$$g=\lambda-\omega=\varepsilon-\pi$$
$$g'=\lambda'-\omega'=\varepsilon'-\pi'$$

the values of x and z become

$$
\begin{aligned}
\frac{x}{a}=&k_1\,(1-\gamma'^2)\cos\{(1+i+j)\,\varepsilon-i\,\pi-(j+j')\,\theta+(i'+j')\,\varepsilon'-i'\,\pi'\}\\
&+k_2\,(1-\gamma'^2)\cos\{(1-i-j)\,\varepsilon+i\,\pi+(j+j')\,\theta-(i'+j')\,\varepsilon'+i'\,\pi'\}\\
&+k_1\,\gamma'^2\cos\{-(1+i+j)\,\varepsilon+i\,\pi+(j+j')\,\theta-(i'+j')\,\varepsilon'+i'\,\pi'+2\,\theta'\}\\
&+k_2\,\gamma'^2\cos\{-(1-i-j)\,\varepsilon-i\,\pi-(j+j')\,\theta+(i'+j')\,\varepsilon'-i'\,\pi'+2\,\theta'\}\\
&-c\,\gamma'\sqrt{1-\gamma'^2}\cos\{(i_1+j_1)\,\varepsilon-i_1\,\pi-(j_1+j'_1)\,\theta+(i'_1+j'_1)\,\varepsilon'-i'_1\,\pi'+\theta'\} \qquad (81)\\
&+c\,\gamma'\sqrt{1-\gamma'^2}\cos\{-(i_1+j_1)\,\varepsilon+i_1\,\pi+(j_1+j'_1)\,\theta-(i'_1+j'_1)\,\varepsilon'+i'_1\,\pi'+\theta'\}\\
\frac{z}{a}=&2\,k_1\,\gamma'\sqrt{1-\gamma'^2}\sin\{(1+i+j)\,\varepsilon-i\,\pi-(j+j')\,\theta+(i'+j')\,\varepsilon'-i'\,\pi'-\theta'\}\\
&+2\,k_2\,\gamma'\sqrt{1-\gamma'^2}\sin\{(1-i-j)\,\varepsilon+i\,\pi+(j+j')\,\theta-(i'+j')\,\varepsilon'+i'\,\pi'-\theta'\}\\
&+c\,(1-2\,\gamma'^2)\sin\{(i_1+j_1)\,\varepsilon-i_1\,\pi-(j_1+j'_1)\,\theta+(i'_1+j'_1)\,\varepsilon'-i'_1\,\pi'\}
\end{aligned}
$$

It is unnecessary to write the value of y, as it is the same with that of x, except that cos is replaced by sin. The entire values of x, y, and z will consist of the sum of an infinite series of terms like the above, formed by the successive substitution of each of the different values of N in the general expressions (79), each term of N giving rise to four terms in x, and each term of N' to two. Should the argument be the same in any two of the terms thus originating they must be combined into one. But it is easy to see that this can not be the case. For, as already shown (§ 8 p. 137), we must then have the coefficients of ε, π, θ, ε', π', and θ' the same in both terms. Now, let us represent any one system of values of i, i', j, j' by

$$i,\ i',\ j,\ j'=s_1,\ s_2,\ s_3,\ s_4,$$

and any other system by

$$i,\ i',\ j,\ j'=t_1,\ t_2,\ t_3,\ t_4.$$

In order that the substitution of these two systems of values in (81) should give the same coefficients of ε, π, etc., in any two terms we must have

$$
\begin{gathered}
s_1+s_3=\pm(t_1+t_3)\\
s_1=\pm t_1\\
s_3+s_4=\pm(t_3+t_4)\\
s_2=\pm t_2
\end{gathered}
$$

the upper signs alone or the lower signs alone being used.

But these equations give

$$s_1=\pm t_1;\quad s_2=\pm t_2;\quad s_3=\pm t_3;\quad s_4=\pm t_4$$

and consequently

$$N_1=\pm N_2$$

But, all the pairs of terms in which the values of N were the same, with opposite signs only, have been combined into one as we went along, so that there are none fulfilling this last condition, as we may easily see by simple inspection of the indices corresponding to the preceding values of k and k'.

Let us represent that part of k_ϵ, k_π, etc., formed from the expressions of the coordinates of the Moon by

$$a\mu_2 h_\epsilon; \quad a\mu_2 h_\pi \quad \text{etc.} \tag{82}$$

Let us also put

p, the coefficient of the time in $i\,g+i'\,g'+j\,\lambda+j'\,\lambda'$, or in N
p_1 that in N′

so that the coefficients of the time in the first four terms of x and y, formerly represented by b, are now $\pm n \pm p$. Referring to the formulæ (62) we see that each value of N and of N′ in the preceding tables gives rise to the following terms in h_ϵ

$$\begin{aligned}
&(1+i+j)\,(n+p)\,k_1^2\,(1-\gamma'^2)^2\\
&+(1-i-j)\,(n-p)\,k_2^2\,(1-\gamma'^2)^2\\
&+(1+i+j)\,(n+p)\,k_1^2\,\gamma'^4\\
&+(1-i-j)\,(n-p)\,k_2^2\,\gamma'^4\\
&+2\,(i_1+j_1)\,p_1\,c^2\,\gamma'^2\,(1-\gamma'^2)\\
&+2\,(1+i+j)\,(n+p)\,k_1^2\,\gamma'^2\,(1-\gamma'^2)\\
&+2\,(1-i-j)\,(n-p)\,k_2^2\,\gamma'^2\,(1-\gamma'^2)\\
&+\frac{1}{2}\,(i_1+j_1)\,p_1\,c^2\,(1-2\,\gamma'^2)^2
\end{aligned}$$

This expression reduces to

$$(1+i+j)\,(n+p)\,k_1^2+(1-i-j)\,(n-p)\,k_2^2+\frac{1}{2}\,(i_1+j_1)\,p_1\,c^2$$

The values of h_π, h_θ, etc., are formed in the same way by replacing $1+i+j$, $1-i-j$, etc., by the corresponding coefficients of π, θ, etc.

Completing the expressions by the sign Σ, which refers to the different systems of values of i, i', j, j', we have

$$\begin{aligned}
h_\epsilon&=\Sigma\left\{(1+i+j)\,(n+p)\,k_1^2+(1-i-j)\,(n-p)\,k_2^2+\frac{1}{2}\,(i_1+j_1)\,p_1\,c^2\right\}\\
h_\pi&=\Sigma\left\{-i(n+p)\,k_1^2+i\,(n-p)\,k_2^2-\frac{1}{2}\,i_1\,p_1\,c^2\right\}\\
h_\theta&=\Sigma\left\{-(j+j')\,(n+p)\,k_1^2+(j+j')\,(n-p)\,k_2^2-\frac{1}{2}\,(j_1+j'_1)\,p_1\,c^2\right\} \qquad (82)\\
h'_\epsilon&=\Sigma\left\{(i'+j')\,(n+p)\,k_1^2-(i'+j')\,(n-p)\,k_2^2+\frac{1}{2}\,(i'_1+j'_1)\,p_1 c^2\right\}\\
h'_\pi&=\Sigma\left\{-i'\,(n+p)\,k_1^2-i'\,(n-p)\,k_2^2-\frac{1}{2}\,i'_1\,p_1\,c^2\right\}\\
h'_\theta&=\Sigma\left\{-2\,(n+p)\,k_1^2-2\,(n-p)\,k_2^2\right\}\gamma'^2
\end{aligned}$$

We now give the expressions of k_1^2, k_2^2, c^2, and p, corresponding to each term of the expressions for $\frac{r}{a}\cos\beta\cos\delta v$, $\frac{r}{a}\cos\beta\sin\delta v$, and $r\sin\beta$, except the first, or constant term of $\frac{r}{a}\cos\beta\cos\delta v$, to which the above expression does not apply. If we consider this term alone, and call it k_0 we shall have

$$h_\iota = n\,k_0^2$$

while all the other values of h will vanish. The formation of k_0^2, k_1^2, k_2^2, is too simple to render any explanation necessary. To form the values of p, it is necessary to have the expressions for the secular motion of the perigee and node of the Moon, or of its mean anomaly and argument of latitude, under the action of the Sun, in terms of the elements of the Sun and Moon. These are obtained from Tome II, p. 237, of **Delaunay**'s Theory, by making the change of elements indicated on p. 800.

We have

$$\begin{aligned}
\frac{dg}{dt} &= n\left\{1-\frac{3}{4}m^2-\frac{225}{32}m^3+\frac{3}{8}e^2m^2-\frac{9}{8}e'^2m^2+6\gamma^2m^2-\frac{4071}{128}m^4\right\}\\
\frac{dg'}{dt} &= mn\\
\frac{d\lambda}{dt} &= n\left\{1+\frac{3}{4}m^2-\frac{9}{32}m^3+\frac{3}{2}e^2m^2+\frac{9}{8}e'^2m^2-\frac{3}{2}\gamma^2m^2-\frac{273}{128}m^4\right\}\\
\frac{d\lambda'}{dt} &= n(m-1)+\frac{d\lambda}{dt}
\end{aligned}\qquad(83)$$

Thus arise the following values of the quantities which enter into the equations (82).

$i,\ i',\ j,\ j'$

0 0 0 0

$$\begin{aligned}
k_0^2 = {} & 1-e^2-2\gamma^2-\frac{1}{3}m^2+\frac{7}{32}e^4+2e^2\gamma^2+\gamma^4-\frac{899}{192}e^2m^2+\frac{37}{96}\gamma^2m^2\\
& -5e'^2m^2+\frac{1591}{1152}m^4-\frac{3315}{64}e^2m^3-\frac{57}{64}\gamma^2m^3+\frac{347}{96}m^5-\frac{5}{144}e^6\\
& +\frac{61}{32}e^4\gamma^2-16e^2\gamma^4-\frac{3497}{768}e^4m^2+\frac{127}{96}\gamma^4m^2-\frac{425}{32}e'^4m^2\\
& +\frac{2567}{96}e^2\gamma^2m^2-\frac{1679}{64}e^2e'^2m^2+\frac{1585}{32}e'^2\gamma^2m^2-\frac{6\,053\,899}{18\,432}e^2m^4\\
& -\frac{60\,439}{9216}\gamma^2m^4+\frac{115\,619}{768}e'^2m^4+\frac{39\,365}{10\,368}m^6-\frac{25}{16}e'^2\alpha^2\\
& -\frac{321}{256}\alpha^2m^2+\frac{63\,651}{64}e'^2m^5-\frac{261}{32}m^7
\end{aligned}$$

$$p=0$$

$i,\quad i',\quad j,\quad j'$

1 0 0 0

$$k_1^2=\frac{1}{4}e^2-\frac{3}{8}e^4-\frac{1}{2}e^2\gamma^2+\frac{7}{24}e^2m^2+\frac{555}{256}e^2m^3+\frac{37}{192}e^6+\frac{1}{8}e^4\gamma^2$$
$$+\frac{3}{2}e^2\gamma^4-\frac{69}{16}e^4m^2-\frac{1673}{384}e^2\gamma^2m^2-\frac{37}{8}e^2e'^2m^2+\frac{75\,083}{9216}e^2m^4$$

$$k_2^2=\frac{9}{4}e^2-\frac{9}{2}e^2\gamma^2-\frac{15}{8}e^2m^2-\frac{1215}{256}e^2m^3+\frac{9}{4}e^2\gamma^4+\frac{1863}{128}e^4m^2$$
$$+\frac{1551}{128}e^2\gamma^2m^2+\frac{45}{2}e^2e'^2m^2-\frac{2131}{1024}e^2m^4$$

$$p=n\left(1-\frac{3}{4}m^2-\frac{225}{32}m^3+\frac{3}{8}e^2m^2-\frac{9}{8}e'^2m^2+6\gamma^2m^2-\frac{4071}{128}m^4\right)$$

2 0 0 0

$$k_1^2=\frac{9}{64}e^4-\frac{9}{32}e^6-\frac{9}{32}e^4\gamma^2+\frac{21}{128}e^4m^2$$

$$k_2^2=\frac{1}{64}e^4+\frac{1}{96}e^6+\frac{9}{32}e^4\gamma^2-\frac{5}{384}e^4m^2$$

$$p=n\left(2-\frac{3}{2}m^2\right)$$

3 0 0 0

$$k_1^2=\frac{1}{9}e^6$$

$$k_2^2=\frac{1}{576}e^6$$

$$p=3n$$

−1 1 0 0

$$k_1^2=\frac{729}{256}e^2e'^2m^2$$

$$k_2^2=\frac{1089}{256}e^2e'^2m^2$$

$$p=-n$$

0 1 0 0

$$k_1^2=\frac{9}{4}e'^2m^2-\frac{9}{4}e'^2m^3+\frac{45}{16}e^2e'^2m^2-\frac{99}{4}e'^2\gamma^2m^2+\frac{81}{16}e'^4m^2$$
$$-\frac{2211}{32}e'^2m^4-\frac{101\,235}{256}e'^2m^5$$

$$k_2^2=\frac{9}{4}e'^2m^2+\frac{9}{4}e'^2m^3+\frac{45}{16}e^2e'^2m^2-\frac{99}{4}e'^2\gamma^2m^2+\frac{81}{16}e'^4m$$
$$-\frac{2211}{32}e'^2m^4-\frac{140\,877}{256}e'^2m^5$$

$$p=mn$$

$i,$	$i',$	$j,$	j'	
1	1	0	0	$k_1^2=\frac{1089}{256}e^2e'^2m^2$
				$k_2^2=\frac{729}{256}e^2e'^2m^2$
				$p=n$
0	2	0	0	$k_1^2=\frac{81}{64}e'^4m^2$
				$k_2^2=\frac{81}{64}e'^4m^2$
				$p=2mn$
-1	0	2	0	$k_1^2=\frac{25}{16}e^2\gamma^4$
				$k_2^2=\frac{81}{16}e^2\gamma^4$
				$p=n$
0	0	2	0	$k_1^2=0$
				$k_2^2=\gamma^4-\frac{9}{4}e^2\gamma^4-\frac{37}{12}\gamma^4m^2$
				$p=n\left(2+\frac{3}{2}m^2\right)$
1	0	2	0	$k_1^2=0$
				$k_2^2=\frac{1}{4}e^2\gamma^4$
				$p=3n$
-2	0	2	-2	$k_1^2=\frac{225}{1024}e^4m^2$
				$k_2^2=\frac{5625}{1024}e^4m^2$
				$p=0$
-1	0	2	-2	$k_1^2=\frac{225}{256}e^2m^2+\frac{5055}{512}e^2m^3-\frac{225}{256}e^4m^2-\frac{675}{128}e^2\gamma^2m^2-\frac{1125}{256}e^2e'^2m^2+\frac{553743}{8192}e^2m^4$

$i \quad i' \quad j \quad j'$

$-1 \quad 0 \quad 2-2$

$$k_2^2=\frac{2025}{256}e^2 m^2+\frac{33\,075}{512}e^2 m^3-\frac{2025}{256}e^4 m^2-\frac{5265}{128}e^2\gamma^2 m^2$$

$$-\frac{10\,125}{256}e^2 e'^2 m^2+\frac{2\,997\,495}{8192}e^2 m^4$$

$$p=n\left(1-2m+\frac{3}{4}m^2\right)$$

$0 \quad 0 \quad 2-2$

$$k_1^2=\frac{9}{256}m^4+\frac{135}{256}e^2 m^3+\frac{21}{64}m^5+\frac{2025}{1024}e^4 m^2+\frac{5085}{1024}e^2 m^4-\frac{27}{128}\gamma^2 m^4$$

$$-\frac{45}{256}e'^2 m^4+\frac{447}{256}m^6-\frac{93}{32}e'^2 m^5+\frac{83}{12}m^7$$

$$k_2^2=\frac{361}{256}m^4-\frac{285}{256}e^2 m^3-\frac{57}{32}\gamma^2 m^3+\frac{1843}{192}m^5-\frac{11\,727}{1024}e^2 m^4-\frac{479}{32}\gamma^2 m^4$$

$$-\frac{1805}{256}e'^2 m^4+\frac{225}{1024}e^4 m^2+\frac{45}{64}e^2\gamma^2 m^2+\frac{9}{16}\gamma^4 m^2+\frac{29\,981}{768}m^6$$

$$-\frac{779}{12}e'^2 m^5+\frac{106\,513}{864}m^7$$

$$p=n(2-2m)$$

$1 \quad 0 \quad 2-2$

$$k_1^2=\frac{9}{64}e^2 m^4$$

$$k_2^2=\frac{1}{256}e^2 m^4$$

$$p=3n$$

$-1 \, -1 \quad 2-2$

$$k_1^2=\frac{1225}{256}e^2 e'^2 m^2$$

$$k_2^2=\frac{11\,025}{256}e^2 e'^2 m^2$$

$$p=n$$

$0 \, -1 \quad 2-2$

$$k_1^2=\frac{441}{1024}e'^2 m^4+\frac{1827}{256}e'^2 m^5$$

$$k_2^2=\frac{17\,689}{1024}e'^2 m^4+\frac{3059}{16}e'^2 m^5$$

$$p=n(2-3m)$$

$-1 \quad 1 \quad 2-2$

$$k_1^2=\frac{225}{256}e^2 e'^2 m^2$$

$$k_2^2=\frac{2025}{256}e^2 e'^2 m^2$$

$$p=n$$

i	i'	j	j'	
0	1	2	−2	$k_1^2=\frac{9}{1024}e'^2m^4+\frac{51}{256}e'^2m^5$ $k_2^2=\frac{361}{1024}e'^2m^4+\frac{1273}{384}e'^2m^5$ $p=n(2-m)$
0	0	0	2	$k_1^2=\frac{9}{4}\gamma^4m^2$ $k_2^2=\frac{9}{16}\gamma^4m^2$ $p=0$
0	0	1	−1	$k_1^2=\frac{225}{1024}\alpha^2m^2$ $k_2^2=\frac{2025}{1024}\alpha^2m^2$ $p=n$
0	1	1	−1	$k_1^2=\frac{25}{64}e'^2\alpha^2$ $k_2^2=\frac{225}{64}e'^2\alpha^2$ $p=n$
−2	0	1	0	$c^2=e^4\gamma^2$ $p_1=-n$
−1	0	1	0	$c^2=9e^2\gamma^2+36e^2\gamma^4-\frac{45}{4}e^4\gamma^2-\frac{1509}{32}e^2\gamma^2m^2$ $p_1=\frac{3}{2}m^2n$
0	0	1	0	$c^2=4\gamma^2-4e^2\gamma^2-4\gamma^4-\frac{4}{3}\gamma^2m^2+\frac{3}{2}\gamma^2m^3-\frac{3}{8}e^4\gamma^2+14e^2\gamma^4$ $+\frac{5}{24}\gamma^4m^2-\frac{17}{12}e^2\gamma^2m^2-2e'^2\gamma^2m^2+\frac{1039}{72}\gamma^2m^4$ $p_1=n\left(1+\frac{3}{4}m^2-\frac{9}{32}m^3+\frac{3}{2}e^2m^2+\frac{9}{8}e'^2m^2-\frac{3}{2}\gamma^2m^2-\frac{273}{128}m^4\right)$

i,	i',	j,	j'	
1	0	1	0	$c^2=e^2\gamma^2-\frac{3}{2}e^4\gamma^2-e^2\gamma^4+\frac{1}{6}e^2\gamma^2m^2$ $p_1=2n$
2	0	1	0	$c^2=\frac{9}{16}e^4\gamma^2$ $p_1=3n$
0	-1	1	0	$c^2=\frac{9}{16}e'^2\gamma^2m^2$ $p_1=n$
0	1	1	0	$c^2=\frac{9}{16}e'^2\gamma^2m^2$ $p_1=n$
-1	0	3	-2	$c^2=\frac{225}{64}e^2\gamma^2m^2$ $p_1=2n$
0	-1	1	-2	$c^2=\frac{49}{16}e'^2\gamma^2m^2$ $p_1=n$
-1	0	1	-2	$c^2=\frac{81}{4}e^2\gamma^2m^2$ $p_1=0$
0	0	1	-2	$c^2=\frac{9}{16}\gamma^2m^2+\frac{123}{32}\gamma^2m^3+\frac{9}{16}\gamma^4m^2+\frac{9}{32}e^2\gamma^2m^2-\frac{45}{16}e'^2\gamma^2m^2$ $+\frac{86655}{512}\gamma^2m^4$ $p_1=n\left(1-2m-\frac{3}{4}m^2\right)$
1	0	1	-2	$c^2=\frac{9}{64}e^2\gamma^2m^2$ $p_1=2n$
0	1	1	-2	$c^2=\frac{9}{16}e'^2\gamma^2m^2$ $p_1=n$

From these expressions we form the values of $\frac{n+p}{n}k_1^2$ and $\frac{n-p}{n}k_2^2$, which, when multiplied by n enter into the expressions (82) for h_ϵ, h_π, etc. The results are shown in the following table. On the left margin are given the indices i, j, i', j', as heretofore; then follow the corresponding expressions for $\frac{n+p}{n}k_1^2$ and $\frac{n-p}{n}k_2^2$ which are given in the middle of the page. For brevity the former is represented by the symbol (1) and the latter by (2). On the right margin are given the coefficients by which these quantities are multiplied in the expressions for h_ϵ, h_π, h_θ, etc. To save space, each set of multipliers is arranged in two lines, and the several sets are separated by a dash.

Each term being multiplied by the corresponding coefficient, and the sums taken, and, in the case of h_ϵ, the value of k_0^2 being also included, we have the values of $\frac{h_\epsilon}{n}$, $\frac{h_\pi}{n}$, etc.

Terms arising from the Longitude.

$i,\ i',\ j,\ j'$		Coefficients for $\epsilon,\ \pi,\ \theta,$ $\epsilon',\ \pi',\ \theta'$
1 0 0 0	$(1)=\frac{1}{2}e^2-\frac{3}{4}e^4-e^2\gamma^2+\frac{19}{48}e^2m^2+\frac{165}{64}e^2m^3+\frac{37}{96}e^6$ $+\frac{1}{4}e^4\gamma^2+3e^2\gamma^4-\frac{33}{4}e^4m^2-\frac{1313}{192}e^2\gamma^2m^2$ $-\frac{305}{32}e^2e'^2m^2+\frac{9359}{1152}e^2m^4$	+2 −1 0 0 0 −2
	$(2)=\frac{27}{16}e^2m^2-\frac{27}{32}e^4m^2+\frac{2025}{128}e^2m^3-\frac{135}{8}e^2\gamma^2m^2$ $+\frac{81}{32}e^2e'^2m^2+\frac{35\,919}{512}e^2m^4$	0 +1 0 0 0 −2
2 0 0 0	$(1)=\frac{27}{64}e^4-\frac{27}{32}e^6-\frac{27}{32}e^4\gamma^2+\frac{9}{32}e^4m^2$	+3 −2 0 0 0 −2
	$(2)=-\frac{1}{64}e^4-\frac{1}{96}e^6-\frac{9}{32}e^4\gamma^2+\frac{7}{192}e^4m^2$	−1 +2 0 0 0 −2
3 0 0 0	$(1)=\frac{4}{9}e^6$	+4 −3 0 0 0 −2
	$(2)=-\frac{1}{288}e^6$	−2 +3 0 0 0 −2
−1 1 0 0	$(1)=0$	
	$(2)=\frac{1089}{128}e^2e'^2m^2$	+2 −1 0 −1 +1 −2

Terms arising from the Longitude—Continued.

$i,\ i',\ j,\ j'$		Coefficients for $\varepsilon,\ \pi,\ \theta,$ $\varepsilon',\ \pi',\ \theta'$
0 1 0 0	$(1)=\frac{9}{4}e'^2m^2+\frac{45}{16}e^2e'^2m^2-\frac{99}{4}e'^2\gamma^2m^2+\frac{81}{16}e'^4m^2-\frac{2283}{32}e'^2m^4-\frac{118\,923}{256}e'^2m^5$	+1 0 0 +1 −1 −2
	$(2)=\frac{9}{4}e'^2m^2+\frac{45}{16}e^2e'^2m^2-\frac{99}{4}e'^2\gamma^2m^2+\frac{81}{16}e'^4m^2-\frac{2283}{32}e'^2m^4-\frac{123\,189}{256}e'^2m^5$	+1 0 0 −1 +1 −2
1 1 0 0	$(1)=\frac{1089}{128}e^2e'^2m^2$	+2 −1 0 +1 −1 −2
	$(2)=0$	
0 2 0 0	$(1)=\frac{81}{64}e'^4m^2$	+1 0 0 +2 −2 −2
	$(2)=\frac{81}{64}e'^4m^2$	+1 0 0 −2 +2 −2
−1 0 2 0	$(1)=\frac{25}{8}e^2\gamma^4$	+2 +1 −2 0 0 −2
	$(2)=0$	
0 0 2 0	$(1)=0$	
	$(2)=-\gamma^4+\frac{9}{4}e^2\gamma^4+\frac{19}{12}\gamma^4m^2$	−1 0 +2 0 0 −2
1 0 2 0	$(1)=0$	
	$(2)=-\frac{1}{2}e^2\gamma^4$	−2 +1 +2 0 0 −2
−2 0 2 −2	$(1)=\frac{225}{1024}e^4m^2$	+1 +2 0 −2 0 −2
	$(2)=\frac{5625}{1024}e^4m^2$	+1 −2 0 +2 0 −2

Terms arising from the Longitude—Continued.

i, i', j, j'		Coefficients for $\varepsilon, \pi, \theta,$ $\varepsilon', \pi', \theta'$
−1 0 2−2	$(1)=\frac{225}{128}e^2m^2+\frac{4605}{256}e^2m^3-\frac{225}{128}e^4m^2-\frac{675}{64}e^2\gamma^2m^2$ $-\frac{1125}{128}e^2e'^2m^2+\frac{475\,563}{4096}e^2m^4$	+2+1 0 −2 0−2
	$(2)=\frac{2025}{128}e^2m^3+\frac{126\,225}{1024}e^2m^4$	0−1 0 +2 0−2
0 0 2−2	$(1)=\frac{27}{256}m^4+\frac{405}{256}e^2m^3+\frac{117}{128}m^5+\frac{6075}{1024}e^4m^2$ $+\frac{14\,175}{1024}e^2m^4-\frac{81}{128}\gamma^2m^4-\frac{135}{256}e'^2m^4+\frac{1173}{256}m^6$ $-\frac{1071}{128}e'^2m^5+\frac{6627}{384}m^7$	+3 0 0 −2 0−2
	$(2)=-\frac{361}{256}m^4+\frac{285}{256}e^2m^3+\frac{57}{32}\gamma^2m^3-\frac{2603}{384}m^5$ $-\frac{225}{1024}e^4m^2-\frac{45}{64}e^2\gamma^2m^2-\frac{9}{16}\gamma^4m^2+\frac{9447}{1024}e^2m^4$ $+\frac{365}{32}\gamma^2m^4+\frac{1805}{256}e'^2m^4-\frac{15\,237}{768}m^6$ $+\frac{19\,513}{384}e'^2m^5-\frac{156\,223}{3456}m^7$	−1 0 0 +2 0−2
1 0 2−2	$(1)=\frac{9}{16}e^2m^4$	+4−1 0 −2 0−2
	$(2)=-\frac{1}{128}e^2m^4$	−2+1 0 +2 0−2
−1−1 2−2	$(1)=\frac{1225}{128}e^2e'^2m^2$	+2+1 0 −3+1−2
	$(2)=0$	
0−1 2−2	$(1)=\frac{1323}{1024}e'^2m^4+\frac{20\,601}{1024}e'^2m^5$	+3 0 0 −3+1−2
	$(2)=-\frac{17\,689}{1024}e'^2m^4-\frac{142\,709}{1024}e'^2m^5$	−1 0 0 +3−1−2

Terms arising from the Longitude—Continued.

$i,\ i',\ j,\ j'$		Coefficients for $\varepsilon,\ \pi,\ \theta;$ $\varepsilon',\ \pi',\ \theta'$
$-1\ \ 1\ \ 2-2$	$(1)=\frac{225}{128}e^2e'^2m^2$	$+2+1\ \ 0$ $-1-1-2$
	$(2)=0$	
$0\ \ 1\ \ 2-2$	$(1)=\frac{27}{1024}e'^2m^4+\frac{603}{1024}e'^2m^5$	$+3\ \ 0\ \ 0$ $-1-1-2$
	$(2)=-\frac{361}{1024}e'^2m^4-\frac{9101}{3072}e'^2m^5$	$-1\ \ 0\ \ 0$ $+1+1-2$
$0\ \ 0\ \ 0\ \ 2$	$(1)=\frac{9}{4}\gamma^4m^2$	$+1\ \ 0-2$ $+2\ \ 0-2$
	$(2)=\frac{9}{16}\gamma^4m^2$	$+1\ \ 0+2$ $-2\ \ 0-2$
$0\ \ 0\ \ 1-1$	$(1)=\frac{225}{512}\alpha^2m^2$	$+2\ \ 0\ \ 0$ $-1\ \ 0-2$
	$(2)=0$	
$0\ \ 1\ \ 1-1$	$(1)=\frac{25}{32}e'^2\alpha^2$	$+2\ \ 0\ \ 0$ $0-1-2$
	$(2)=0$	

Terms arising from the Latitude,

or values of $\frac{1}{2}\frac{p}{n}c^2$

$i,\ i',\ j,\ j'$		Coefficients for $\varepsilon,\ \pi,\ \theta$ $\varepsilon',\ \pi',\ \theta'$
$-2\ \ 0\ \ 1\ \ 0$	$-\frac{1}{2}e^4\gamma^2$	$-1+2-1$ $0\ \ 0\ \ 0$
$-1\ \ 0\ \ 1\ \ 0$	$\frac{27}{4}e^2\gamma^2m^2$	$0+1-1$ $0\ \ 0\ \ 0$

Terms arising from the Latitude,

or values of $\frac{1}{2}\frac{p}{n}c^2$—Continued.

i,	i',	j,	j'		Coefficients for ε, π θ ε', π', θ'
0	0	1	0	$2\gamma^2-2e^2\gamma^2-2\gamma^4+\frac{5}{6}\gamma^2m^2+\frac{3}{16}\gamma^2m^3-\frac{3}{16}e^4\gamma^2$ $+7e^2\gamma^4+\frac{19}{24}e^2\gamma^2m^2+\frac{5}{4}e'^2\gamma^2m^2-\frac{211}{48}\gamma^4m^2$ $+\frac{1411}{576}\gamma^2m^4$	$+1\ \ 0-1$ $0\ \ 0\ \ 0$
1	0	1	0	$e^2\gamma^2-\frac{3}{2}e^4\gamma^2-e^2\gamma^4+\frac{1}{6}e^2\gamma^2m^2$	$+2-1-1$ $0\ \ 0\ \ 0$
2	0	1	0	$\frac{27}{32}e^4\gamma^2$	$+3-2-1$ $0\ \ 0\ \ 0$
0	−1	1	0	$\frac{9}{32}e'^2\gamma^2m^2$	$+1\ \ 0-1$ $-1+1\ \ 0$
0	1	1	0	$\frac{9}{32}e'^2\gamma^2m^2$	$+1\ \ 0-1$ $+1-1\ \ 0$
−1	0	3	−2	$\frac{225}{64}e^2\gamma^2m^2$	$+2+1-1$ $-2\ \ 0\ \ 0$
0	0	3	−2	$\frac{27}{128}\gamma^2m^4$	$+3\ \ 0-1$ $-2\ \ 0\ \ 0$
0	−1	1	−2	$\frac{49}{32}e'^2\gamma^2m^2$	$+1\ \ 0+1$ $-3+1\ \ 0$
−1	0	1	−2	0	$0+1+1$ $-2\ \ 0\ \ 0$
0	0	1	−2	$\frac{9}{32}\gamma^2m^2+\frac{87}{64}\gamma^2m^3+\frac{9}{64}e^2\gamma^2m^2-\frac{45}{32}e'^2\gamma^2m^2$ $+\frac{9}{32}\gamma^4m^2+\frac{4503}{1024}\gamma^2m^4$	$+1\ \ 0+1$ $-2\ \ 0\ \ 0$
1	0	1	−2	$\frac{9}{64}e^2\gamma^2m^2$	$+2-1+1$ $-2\ \ 0\ \ 0$
0	1	1	−2	$\frac{9}{32}e'^2\gamma^2m^2$	$+1\ \ 0+1$ $-1-1\ \ 0$

We have now all the data for forming, by means of the equations (82), the final values of h_ϵ, h_π, etc., so far as they depend on the coordinates x, y, z of the Moon, which parts are represented below by adding the symbol ☽. We shall, however, for convenience in our subsequent work, change the form of the common factor $\mu_2 a^2 n$ by which all these quantities are multiplied in order to obtain k_ϵ, k_π, etc. We recall that, in DELAUNAY's theory, the quantities a and n are connected by the relation

$$a^3 n^2 = m_2 + m_3$$

Then, since we have put

$$\mu_2 = \frac{m_2 m_3}{m_2 + m_3}$$

we have

$$\mu_2 a^2 n = \frac{m_2 m_3}{a n}$$

which we shall adopt for the common factor in question.

We now find from the preceding tables for the terms in question.

$$\begin{aligned}
k_{\epsilon ☽} &= \frac{m_2 m_3}{a n}\Big\{1 + \Big(-\frac{1}{3} - \frac{3}{8}e^2 - \frac{1}{2}e'^2 + \frac{3}{2}\gamma^2 - \frac{1}{32}e^4 - \frac{9}{16}e^2 e'^2 - \frac{5}{8}e'^4 + \frac{5}{4}e^2\gamma^2 + \frac{9}{4}e'^2\gamma^2 \\
&\qquad - \gamma^4 - \frac{3}{8}\alpha^2\Big)m^2 \\
&\quad + \Big(-\frac{225}{32}e^2 - \frac{9}{8}\gamma^2\Big)m^3 \\
&\quad + \Big(\frac{895}{288} - \frac{23\,255}{512}e^2 + \frac{3995}{192}e'^2 - \frac{1585}{128}\gamma^2\Big)m^4 \\
&\quad + \Big(\frac{1261}{96} + \frac{11\,349}{64}e'^2\Big)m^5 + \frac{3028}{81}m^6 + \frac{76\,741}{864}m^7\Big\} \\
k_{\pi ☽} &= \frac{m_2 m_3}{a n}\Big\{-\frac{1}{2}e^2 - \frac{1}{8}e^4 + \frac{5}{8}e^2\gamma^4 - \frac{5}{16}e^4\gamma^2 - \frac{1}{16}e^6 \\
&\quad + \Big(\frac{1171}{384}e^2 - \frac{2069}{384}e^4 - \frac{309}{128}e^2 e'^2 - \frac{85}{8}e^2\gamma^2\Big)m^2 \\
&\quad + \frac{3945}{256}e^2 m^3 + \frac{2\,001\,623}{36\,864}e^2 m^4\Big\} \\
k_{\theta ☽} &= \frac{m_2 m_3}{a n}\Big\{-2\gamma^2 + e^2\gamma^2 - \frac{35}{4}e^2\gamma^4 + \frac{43}{32}e^4\gamma^2 \\
&\quad + \Big(-\frac{53}{96}\gamma^2 - \frac{2101}{192}e^2\gamma^2 - \frac{45}{32}e'^2\gamma^2 + \frac{143}{32}\gamma^4\Big)m^2 \\
&\quad + \frac{75}{64}\gamma^2 m^3 + \frac{16\,007}{9216}\gamma^2 m^4\Big\}
\end{aligned} \tag{84}$$

$$\begin{aligned}
k_{\epsilon'\rightmoon} &= -\frac{m_2\, m_3}{a\, n}\left\{\left(\frac{225}{64}\, e^2 + \frac{9}{16}\, \gamma^2\right) m^2 - \frac{97}{32} m^4 + \text{etc.}\right\} \\
k_{\pi'\rightmoon} &= +\frac{m_2\, m_3}{a\, n}\left\{\left(\frac{125}{16}\, e^2\, e'^2 + \frac{5}{4}\, e'^2\, \gamma^2\right) m^2 + \frac{291}{16}\, e'^2\, m^4 \text{etc.}\right\} \qquad (85) \\
k_{\theta'\rightmoon} &= +\frac{m_2\, m_3}{a\, n}\left\{-2\, \gamma'^2 + 4\, \gamma^2\, \gamma'^2 + e^2\, \gamma'^2 + \frac{2}{3}\, \gamma'^2\, m^2 + \text{etc.}\right\}
\end{aligned}$$

We have next to consider the corresponding quantities formed from the solar coordinates, x', y', z'. Owing to the minutenesss of the perturbations of these coordinates in the problem of three bodies, the largest terms of this kind will be those arising from the values of x', y', and z' in the theory of elliptic motion. Into this theory the lunar elements g, π, and θ do not enter; it will not therefore add any terms to k_ϵ, k_π or k_θ. However from the form of the canonical equations (64) it appears that k_ϵ', k_π', and k_θ' are the conjugate elements of ϵ', π', and θ', that is to say, functions of ϵ', π', and θ' satisfying the conditions

$$[\epsilon', k_\epsilon'] = 1; \quad [\pi', k_\pi'] = 1; \quad [\theta', k_\theta'] = 1$$

while all other combinations of the same kind vanish. Now in the theory of elliptic motion the values of these conjugate functions are known to be

$$\begin{aligned}
k_\epsilon'{}_\odot &= m_1 \frac{(m_2 + m_3)}{a'\, n'} \\
k_\pi'{}_\odot &= k_\epsilon\, (1 - e'^2)^{\frac{1}{2}} \qquad (86) \\
k_\theta'{}_\odot &= k_\pi' \cos i' = k_\pi'\, (1 - 2\, \gamma'^2)
\end{aligned}$$

which are therefore the principal terms in question.

We have next to consider the order of magnitude of the terms which will be introduced into the values of x', y', and z' by the perturbations of these quantities due to the separation of the Earth and Moon. From the process of integration developed in §§ 3, 5, and 6 when Ω_1 is taken account of, it follows that any term of x' in (60) which contains either ϵ, π, or θ in its argument will contain the factor

$$k = \mu \frac{a^2}{a'^2}$$

and will be of the form

$$x' = k\, m\, s\, a' \cos\, (\mathrm{I}\, \epsilon + \mathrm{I}^1\, \pi + \quad . \quad . \quad .)$$

s being a numerical factor of the order of magnitude unity or less. The principal term of k_ϵ thus arising will be by (62)

$$k_{\epsilon\odot} = \mu_1\, \mathrm{I}^2\, n\, k^2\, m^2\, s^2\, a'^2$$

and the complete expression for $k_{\epsilon\odot}$ will contain the factor $\mu_1\, n\, k^2\, m^2\, a'^2$ the ratio of which to the common factor $\mu_2\, a^2\, n$ of $k_{\epsilon\rightmoon}$ is of the evanescent order of magnitude $m^2\, k$. These terms are, therefore, entirely unimportant, and the complete values of k_ϵ, k_π, and k_θ may be taken as equal to $k_{\epsilon\rightmoon}$, etc.

The terms of x' which depend on the arguments ϵ', π', θ' exclusively will receive increments of the order of magnitude of the terms in ϵ, π, and θ; that is to say, the terms in question will be of the form

$$x' = a'(s' + k\,m\,s)\cos(i_4\,\varepsilon' + i_5\,\pi' + i_6\,\theta')$$

s' and s being numerical factors. Each term of this kind will give rise in $k_\epsilon'{}_\odot$ to the term

$$i_4^2\,a'^2\,n'\,(s' + k\,m\,s)^2$$

The principal term $i_4^2\,a'^2\,n'\,s'^2$ of this expression will be contained in the value of $k_\epsilon'{}_\odot$ already given in (86). The principal part of the remaining terms will have as a factor the factor of the former terms, $\mu_1\,a'^2\,n'$, multiplied by $m\,k$. This gives rise to the common factor

$$\frac{m_2\,m_3}{a\,n}m^2$$

in the terms of k_ϵ', k_π', and k_θ'. The terms introduced by the coordinates x, y, and, z, as found in (85), contain this same factor. We conclude, therefore, that this portion of k_ϵ' etc., must remain undetermined until the solution of the problem of three bodies is complete.

§ 12.

FORMATION OF THE DIFFERENTIAL EQUATIONS.

We have now to differentiate the preceding values of k_ϵ, k_θ, etc., with respect to the elements of which they are functions. To find the derivatives of the powers of m with respect to the logarithm of a, which I call x, we remark that its value is

$$m = \frac{n'}{n} = \left\{\frac{m_1 + m_2 + m_3}{m_2 + m_3}\,\frac{a^3}{a'^3}\right\}^{\frac{1}{2}}$$

From this we derive the general rule that if

$$u = a^i\{c_0 + c_1\,m + c_2\,m^2 + c_3\,m^3 + \text{etc.}\}$$

then

$$\left.\begin{aligned}\frac{d\,u}{d\,x} &= a^i\left\{i\,c_0 + \left(i + \frac{3}{2}\right)c_1\,m + \left(i + \frac{6}{2}\right)c_2\,m^2 + \text{etc.}\right\}\\ \frac{d\,u}{d\,x'} &= a^i\left\{-\frac{3}{2}c_1\,m - \frac{6}{2}c_2\,m^2 - \frac{9}{2}c_3\,m^3 - \text{etc.}\right\}\end{aligned}\right\} \qquad (87)$$

We thus find, dropping the symbols ☽ and ⊙ in (84) and (86),

$$\begin{aligned}\frac{d\,k_\epsilon}{d\,x} = \frac{m_2\,m_3}{a\,n}\Big\{&\frac{1}{2} + \Big(-\frac{7}{6} - \frac{21}{16}e^2 + \frac{21}{4}\gamma^2 - \frac{7}{4}e'^2 - \frac{7}{64}e^4 + \frac{35}{8}e^2\gamma^2 - \frac{7}{2}\gamma^4 - \frac{63}{32}e^2e'^2 + \frac{63}{8}e'^2\gamma^2\\ &- \frac{35}{16}e'^4 - \frac{33}{16}\alpha^2\Big)m^2 + \Big(-\frac{1125}{32}e^2 - \frac{45}{8}\gamma^2\Big)m^3 + \Big(\frac{11\,635}{576} - \frac{302\,315}{1024}e^2\\ &- \frac{20\,605}{256}\gamma^2 + \frac{51\,935}{384}e'^2\Big)m^4 + \Big(\frac{1261}{12} + \frac{11\,349}{8}e'^2\Big)m^5 + \frac{28\,766}{81}m^6\\ &+ \frac{844\,151}{864}m^7\Big\}\end{aligned}$$

$$\frac{d\,k_\epsilon}{d\,e}=\frac{m_2m_3}{a\,n}\left\{\left(-\frac{3}{4}e-\frac{1}{8}e^3+\frac{5}{2}e\,\gamma^2-\frac{9}{8}e\,e'^2\right)m^2-\frac{225}{16}e\,m^3-\frac{23\,255}{256}e\,m^4\right\}$$

$$\frac{d\,k_\epsilon}{d\,\gamma}=\frac{m_2m_3}{a\,n}\left\{\left(3\,\gamma+\frac{5}{2}e^2\gamma-4\,\gamma^3+\frac{9}{2}e'^2\gamma\right)m^2-\frac{9}{4}\gamma\,m^3-\frac{1585}{64}\gamma\,m^4\right\}$$

$$\frac{d\,k_\epsilon}{d\,e'}=\frac{m_2m_3}{a\,n}\left\{\left(-e'-\frac{9}{8}e^2e'+\frac{9}{2}\gamma^2e'-\frac{5}{2}e'^3\right)m^2+\frac{3995}{96}e'\,m^4+\frac{11\,349}{32}e'\,m^5\right\}$$

$$\frac{d\,k_\pi}{d\,\varpi}=\frac{m_2m_3}{a\,n}\left\{-\frac{1}{4}e^2-\frac{1}{16}e^4-\frac{1}{32}e^6-\frac{5}{32}e^4\gamma^2+\frac{5}{16}e^2\gamma^4+\left(\frac{8197}{768}e^2-\frac{14\,483}{768}e^4-\frac{595}{16}e^2\gamma^2\right.\right.$$
$$\left.\left.-\frac{2163}{256}e^2e'^2\right)m^2+\frac{19\,725}{256}e^2m^3+\frac{26\,021\,099}{73\,728}e^2m^4\right\}$$

$$\frac{d\,k_\pi}{d\,e}=\frac{m_2m_3}{a\,n}\left\{-e-\frac{1}{2}e^3+\frac{5}{4}e\,\gamma^4-\frac{5}{4}e^3\gamma^2-\frac{3}{8}e^5+\left(\frac{1171}{192}e-\frac{2069}{96}e^3-\frac{85}{4}e\,\gamma^2-\frac{309}{64}e\,e'^2\right)m^2\right.$$
$$\left.+\frac{3945}{128}e\,m^3+\frac{2\,001\,623}{18\,432}e\,m^4\right\}$$

$$\frac{d\,k_\pi}{d\,\gamma}=\frac{m_2m_3}{a\,n}\left\{\frac{5}{2}e^2\gamma^3-\frac{5}{8}e^4\gamma-\frac{85}{4}e^2\gamma\,m^2\right\}$$

$$\frac{d\,k_\pi}{d\,e'}=\frac{m_2m_3}{a\,n}\left\{-\frac{309}{64}e^2e'\,m^2\right\}$$

$$\frac{d\,k_\theta}{d\,\varpi}=\frac{m_2m_3}{a\,n}\left\{-\gamma^2+\frac{1}{2}e^2\gamma^2-\frac{35}{8}e^2\gamma^4-\frac{43}{64}e^4\gamma^2+\left(-\frac{371}{192}\gamma^2-\frac{14\,707}{384}e^2\gamma^2+\frac{1001}{64}\gamma^4\right.\right.$$
$$\left.\left.-\frac{315}{64}e'^2\gamma^2\right)m^2+\frac{375}{64}\gamma^2m^3+\frac{208\,091}{18\,432}\gamma^2m^4\right\}$$

$$\frac{d\,k_\theta}{d\,e}=\frac{m_2m_3}{a\,n}\left\{2\,e\,\gamma^2-\frac{35}{2}e\,\gamma^4+\frac{43}{8}e^3\gamma^2-\frac{2101}{96}e\,\gamma^2m^2\right\}$$

$$\frac{d\,k_\theta}{d\,\gamma}=\frac{m_2m_3}{a\,n}\left\{-4\,\gamma+2\,e^2\gamma-35\,e^2\gamma^3+\frac{43}{16}e^4\gamma+\left(-\frac{53}{48}\gamma-\frac{2101}{96}e^2\gamma+\frac{143}{8}\gamma^3\right.\right.$$
$$\left.-\frac{45}{16}e'^2\gamma\right)m^2+\frac{75}{32}\gamma\,m^3+\frac{16\,007}{4608}\gamma\,m^4$$

$$\frac{d\,k_\theta}{d\,e'}=\frac{m_2m_3}{a\,n}\left\{-\frac{45}{16}\gamma^2e'\,m^2\right\}$$

The derivatives with respect to ϖ' may be found by the condition

$$\frac{d\,k}{d\,\varpi}+\frac{d\,k}{d\,\varpi'}=\frac{1}{2}k$$

A rigorous solution of the problem will now be obtained by taking the derivatives of k_ϵ', k_π', and k_θ', with respect to the six elements which they contain, substituting all the derivatives in general equations of the form (64), and then solving these equations with respect to $\frac{d\,\varpi}{d\,t}$, $\frac{d\,e}{d\,t}$, etc. But we have seen that in the values of k_ϵ', k_θ', etc., the lunar elements are multiplied by factors so minute that they may be wholly neglected. The perturbations of the solar elements may therefore be obtained from the values of k_ϵ', k_π' and k_θ', and the results substituted in the expressions for the lunar elements. The process will be as follows:

From the equations

$$\begin{aligned} \frac{dx'}{dt} &= \frac{dx'}{dk_\epsilon'}\frac{dR}{d\epsilon'} + \frac{dx'}{dk_\pi'}\frac{dR}{d\pi'} + \frac{dx'}{dk'_\theta}\frac{dR}{d\theta'} \\ \frac{de'}{dt} &= \frac{de'}{dk'_\epsilon}\frac{dR}{d\epsilon'} + \frac{de'}{dk'_\pi}\frac{dR}{d\pi'} + \frac{de'}{dk'_\theta}\frac{dR}{d\theta'} \end{aligned} \tag{88}$$

we are to derive the perturbations of x' and e'. Those of γ' are at present omitted, because this quantity does not enter into k_ϵ, k_π, or k_θ, so far as we have written them.

From the equations

$$\begin{aligned} \frac{d\epsilon'}{dt} &= n' - \frac{dx'}{dk_\epsilon'}\frac{dR}{dx'} - \frac{de'}{dk_\epsilon'}\frac{dR}{de'} - \frac{d\gamma'}{dk_\epsilon'}\frac{dR}{d\gamma'} \\ \frac{d\pi'}{dt} &= -\frac{dx'}{dk_\pi'}\frac{dR}{dx'} - \frac{de'}{dk_\pi'}\frac{dR}{de'} - \frac{d\gamma'}{dk_\epsilon'}\frac{dR'}{d\gamma'} \end{aligned}$$

where

$$n' = \{m_1+m_2+m_3\}^{\frac{1}{2}}\, a'^{-\frac{3}{2}} = (m_1+m_2+m_3)^{\frac{1}{2}}\, c^{-\frac{3}{2}x'}$$

c being the Neperien base, we are to derive ϵ' and π', or their perturbations.

Putting the first three of the equations (64) in the form

$$\begin{aligned} \frac{dk_\epsilon}{dx}\frac{dx}{dt} + \frac{dk_\epsilon}{de}\frac{de}{dt} + \frac{dk_\epsilon}{d\gamma}\frac{d\gamma}{dt} &= \frac{dR}{d\epsilon} - \frac{dk_\epsilon}{dx'}\frac{dx'}{dt} - \frac{dk_\epsilon}{de'}\frac{de'}{dt} \\ \frac{dk_\pi}{dx}\frac{dx}{dt} + \frac{dk_\pi}{de}\frac{de}{dt} + \frac{dk_\pi}{d\gamma}\frac{d\gamma}{dt} &= \frac{dR}{d\pi} - \frac{dk_\pi}{dx'}\frac{dx'}{dt} - \frac{dk_\pi}{de'}\frac{de'}{dt} \\ \frac{dk_\theta}{dx}\frac{dx}{dt} + \frac{dk_\theta}{de}\frac{de}{dt} + \frac{dk_\theta}{d\gamma}\frac{d\gamma}{dt} &= \frac{dR}{d\theta} - \frac{dk_\theta}{dx'}\frac{dx'}{dt} - \frac{dk_\theta}{de'}\frac{de'}{dt} \end{aligned} \tag{89}$$

We are to solve these equations with respect to $\frac{dx}{dt}, \frac{de}{dt}$, and $\frac{d\gamma}{dt}$. Let us put

t', the time introduced through the varying solar elements, so that

$$\frac{dk_\epsilon}{dt'} = \frac{dk_\epsilon}{dx'}\frac{dx'}{dt} + \frac{dk_\epsilon}{de'}\frac{de'}{dt}$$

etc. etc. etc.

We shall then have

$$\begin{aligned}
\frac{dv}{dt} &= \left(\frac{dv}{dk_\epsilon}\right) \cdot \left(\frac{dR}{d\epsilon} - \frac{dk_\epsilon}{dt'}\right) \\
&+ \left(\frac{dv}{dk_\pi}\right) \cdot \left(\frac{dR}{d\pi} - \frac{dk_\pi}{dt'}\right) \\
&+ \left(\frac{dv}{dk_\theta}\right) \cdot \left(\frac{dR}{d\theta} - \frac{dk_\theta}{dt'}\right) \\
\frac{de}{dt} &= \left(\frac{de}{dk_\epsilon}\right) \cdot \left(\frac{dR}{d\epsilon} - \frac{dk_\epsilon}{dt'}\right) \\
&+ \left(\frac{de}{dk_\pi}\right) \cdot \left(\frac{dR}{d\pi} - \frac{dk_\pi}{dt'}\right) \\
&+ \left(\frac{de}{dk_\theta}\right) \cdot \left(\frac{dR}{d\theta} - \frac{dk_\theta}{dt'}\right) \\
\frac{d\gamma}{dt} &= \left(\frac{d\gamma}{dk_\epsilon}\right) \cdot \left(\frac{dR}{d\epsilon} - \frac{dk_\epsilon}{dt'}\right) \\
&+ \left(\frac{d\gamma}{dk_\pi}\right) \cdot \left(\frac{dR}{d\pi} - \frac{dk_\pi}{dt}\right) \\
&+ \left(\frac{d\gamma}{dk_\theta}\right) \cdot \left(\frac{dR}{d\theta} - \frac{dk_\pi}{dt'}\right)
\end{aligned} \tag{90}$$

The derivatives $\left(\frac{dv}{dk_\epsilon}\right)$, etc., are here inclosed in parentheses to indicate that they are to be taken subject to the condition that v' and e' shall remain constant; in other words, the derivatives with respect to k_ϵ are to be formed by solving the equations

$$\begin{aligned}
\frac{dk_\epsilon}{dv}\left(\frac{dv}{dk_\epsilon}\right) + \frac{dk_\epsilon}{de}\left(\frac{de}{dk_\epsilon}\right) + \frac{dk_\epsilon}{d\gamma}\left(\frac{d\gamma}{dk_\epsilon}\right) &= 1 \\
\frac{dk_\pi}{dv}\left(\frac{dv}{dk_\epsilon}\right) + \frac{dk_\pi}{de}\left(\frac{de}{dk_\epsilon}\right) + \frac{dk_\pi}{d\gamma}\left(\frac{d\gamma}{dk_\epsilon}\right) &= 0 \\
\frac{dk_\theta}{dv}\left(\frac{dv}{dk_\epsilon}\right) + \frac{dk_\theta}{de}\left(\frac{de}{dk_\epsilon}\right) + \frac{dk_\theta}{d\gamma}\left(\frac{d\gamma}{dk_\epsilon}\right) &= 0
\end{aligned} \tag{91}$$

and those with respect to k_π and k_θ by solving similar sets in which those quantities enter. Having found the perturbations of v, e, and γ, those of ϵ, π, and θ are to be obtained from the equations

$$\frac{d\,k_\epsilon}{d\,x}\frac{d\,\epsilon_0}{d\,t}+\frac{d\,k_\pi}{d\,x}\frac{d\,\pi_0}{d\,t}+\frac{d\,k_\theta}{d\,x}\frac{d\,\theta_0}{d\,t}=-\frac{d\,\mathrm{R}}{d\,x}$$

$$\frac{d\,k_\epsilon}{d\,e}\frac{d\,\epsilon_0}{d\,t}+\frac{d\,k_\pi}{d\,e}\frac{d\,\pi_0}{d\,t}+\frac{d\,k_\theta}{d\,e}\frac{d\,\theta_0}{d\,t}=-\frac{d\,\mathrm{R}}{d\,e}$$

$$\frac{d\,k_\epsilon}{d\,\gamma}\frac{d\,\epsilon_0}{d\,t}+\frac{d\,k_\pi}{d\,\gamma}\frac{d\,\pi_0}{d\,t}+\frac{d\,k_\theta}{d\,\gamma}\frac{d\,\theta_0}{d\,t}=-\frac{d\,\mathrm{R}}{d\,\gamma}$$

which give

$$\begin{aligned}
\frac{d\,\epsilon}{d\,t}&=\frac{d\,\epsilon_0}{d\,t}+n\;=n\;-\left(\frac{d\,x}{d\,k_\epsilon}\right)\frac{d\,\mathrm{R}}{d\,x}-\left(\frac{d\,e}{d\,k_\epsilon}\right)\frac{d\,\mathrm{R}}{d\,e}-\left(\frac{d\,\gamma}{d\,k_\epsilon}\right)\frac{d\,\mathrm{R}}{d\,\gamma}\\
\frac{d\,\pi}{d\,t}&=\frac{d\,\pi_0}{d\,t}+\pi_1=\pi_1-\left(\frac{d\,x}{d\,k_\pi}\right)\frac{d\,\mathrm{R}}{d\,x}-\left(\frac{d\,e}{d\,k_\pi}\right)\frac{d\,\mathrm{R}}{d\,e}-\left(\frac{d\,\gamma}{d\,k_\pi}\right)\frac{d\,\mathrm{R}}{d\,\gamma}\\
\frac{d\,\theta}{d\,t}&=\frac{d\,\theta_0}{d\,t}+\theta_1=\theta_1-\left(\frac{d\,x}{d\,k_\theta}\right)\frac{d\,\mathrm{R}}{d\,x}-\left(\frac{d\,e}{d\,k_\theta}\right)\frac{d\,\mathrm{R}}{d\,e}-\left(\frac{d\,\gamma}{d\,k_\theta}\right)\frac{d\,\mathrm{R}}{d\,\gamma}
\end{aligned}\qquad(92)$$

The values of n, π_1, and θ_1 are the expressions for the mean motion and the secular motions of the perigee and the node in the theory of three bodies, which last are taken from DELAUNAY (*Comptes Rendus*, 1872, I, Vol. 74, p. 19), as follows:

$$n=(m_2+m_3)^{\frac{1}{2}}\,c^{-\frac{3}{2}}$$

$$\begin{aligned}
\pi_1=m^2n\Big\{&\frac{3}{4}-\frac{3}{8}e^2-6\gamma^2+\frac{9}{8}e'^2-\frac{3}{32}e^4+\frac{69}{8}e^2\gamma^2-\frac{45}{4}\gamma^4-9\gamma^2e'^2-\frac{9}{16}e^2e'^2+\frac{45}{32}e'^4\\
&+\left(\frac{225}{32}-\frac{675}{64}e^2-\frac{189}{8}\gamma^2+\frac{825}{32}e'^2+\frac{81}{32}e^2\gamma^2+\frac{1107}{16}\gamma^4-\frac{349}{4}\gamma^2e'^2-\frac{2475}{64}e^2e'^2\right)m\\
&+\left(\frac{4071}{128}-\frac{31\,605}{512}e^2-\frac{3963}{32}\gamma^2+\frac{61\,179}{256}e'^2\right)m^2\\
&+\left(\frac{265\,493}{2048}-\frac{1\,483\,665}{4096}e^2-\frac{335\,403}{512}\gamma^2+\frac{1\,767\,849}{1024}e'^2\right)m^3\\
&+\frac{12\,822\,631}{24\,576}m^4+\frac{1\,273\,925\,965}{589\,824}m^5+\frac{71\,028\,685\,589}{7\,077\,888}m^6\Big\}
\end{aligned}$$

$$\begin{aligned}
\theta_1=m^2n\Big\{&-\frac{3}{4}-\frac{3}{2}e^2+\frac{3}{2}\gamma^2-\frac{9}{8}e'^2+\frac{21}{64}e^4-\frac{51}{8}e^2\gamma^2+\frac{9}{4}e'^2\gamma^2-\frac{9}{4}e^2e'^2-\frac{45}{32}e'^4\\
&+\left(\frac{9}{32}-\frac{189}{32}e^2-\frac{27}{16}\gamma^2+\frac{23}{32}e'^2-\frac{675}{256}e^4+\frac{567}{16}e^2\gamma^2+\frac{27}{16}\gamma^4-\frac{349}{16}e^2e'^2-\frac{99}{16}\gamma^2e'^2\right)m\\
&+\left(\frac{273}{128}-\frac{2739}{128}e^2-\frac{843}{128}\gamma^2+\frac{3261}{256}e'^2\right)m^2\\
&+\left(\frac{9797}{2048}-\frac{165\,411}{2048}e^2-\frac{7185}{1024}\gamma^2+\frac{73\,423}{1024}e'^2\right)m^3+\frac{199\,273}{24\,576}m^4+\frac{6\,657\,733}{589\,824}m^5\Big\}
\end{aligned}$$

The perturbations of ν, e, and γ, which are of the first order as to the disturbing forces, and found by integrating (90) as if the elements were constant in the second members, may be divided into two classes: (1) Those which arise from the derivatives of R with respect to ε, π, and θ, and (2) those which arise from $\frac{d\,k_\varepsilon}{d\,t'}$, etc. Let us consider the latter. They are given by the integration of the equations

$$\begin{aligned}
\frac{d\nu}{d\,t} &= -\left(\frac{d\,\nu}{d\,k_\varepsilon}\right)\frac{d\,k_\varepsilon}{d\,t'} - \left(\frac{d\,\nu}{d k_\pi}\right)\frac{d\,k_\pi}{d\,t'} - \left(\frac{d\,\nu}{d\,k_\theta}\right)\frac{d\,k_\theta}{d\,t'} \\
\frac{d\,e}{d\,t} &= -\left(\frac{d\,e}{d\,k_\varepsilon}\right)\frac{d k_\varepsilon}{d\,t'} - \left(\frac{d\,e}{d\,k_\pi}\right)\frac{d\,k_\pi}{d\,t'} - \left(\frac{d\,e}{d\,k_\theta}\right)\frac{d\,k_\theta}{d\,t'} \\
\frac{d\gamma}{d\,t} &= -\left(\frac{d\gamma}{d\,k_\varepsilon}\right)\frac{d\,k_\varepsilon}{d\,t'} - \left(\frac{d\,\gamma}{d\,k_\pi}\right)\frac{d\,k_\pi}{d\,t'} - \left(\frac{d\,\gamma}{d k_\theta}\right)\frac{d\,k_\theta}{d\,t'}
\end{aligned} \qquad (93)$$

The corresponding perturbations, $\delta\,\nu$, $\delta\,e$, and $\delta\,\gamma$, to quantities of the first order will be given by solving the equations

$$\begin{aligned}
\frac{d\,k_\varepsilon}{d\nu}\delta\,\nu + \frac{d\,k_\varepsilon}{d\,e}\delta\,e + \frac{d\,k_\varepsilon}{d\gamma}\delta\,\gamma + \frac{d\,k_\varepsilon}{d\nu'}\delta\,\nu' + \frac{d\,k_\varepsilon}{d\,e'}\delta\,e' &= 0 \\
\frac{d\,k_\pi}{d\,\nu}\delta\,\nu + \frac{d\,k_\pi}{d\,e}\delta\,e + \frac{d\,k_\pi}{d\,\gamma}\delta\,\gamma + \frac{d\,k_\pi}{d\nu'}\delta\,\nu' + \frac{d\,k_\pi}{d\,e'}\delta\,e' &= 0 \\
\frac{d\,k_\theta}{d\nu}\delta\,\nu + \frac{d\,k_\theta}{d\,e}\delta\,e + \frac{d\,k_\theta}{d\,\gamma}\delta\,\gamma + \frac{d\,k_\theta}{d\,\nu'}\delta\,\nu' + \frac{d\,k_\theta}{d\,e'}\delta\,e' &= 0
\end{aligned} \qquad (94)$$

The values of $\delta\,\nu'$ and $\delta\,e'$ are regarded as known in advance. This curious theorem may embody some principle applicable to the disturbed motion of three bodies, which has not yet been fully mastered. The quantities I have called k_ε, k_π, and k_θ, are closely related to DELAUNAY'S L, G, and H, as is shown in the next section. The theorem shows that the secular acceleration of the mean motion and of the motion of the perigee and nodes may be found by the equations

$$\delta\,\mathrm{L} = 0; \quad \delta\,\mathrm{G} = 0; \quad \delta\,\mathrm{H} = 0;$$

by making the transformation on p. 800 of DELAUNAY'S *Theorie*, Vol. II. This result I have verified so far as the mean motion is concerned.

The numerical values of the derivatives of k_ϵ, k_π, and k_θ from the formulæ of pp. 186–87 are next to be computed. To exhibit the uncertainty to which many of the derivatives are subject, owing to the approximation in powers of m, γ, e, and e' not being carried far enough, the results are arranged by powers of m, the quantity whose powers converge least rapidly. From the last two or three terms the probable law of formation is roughly guessed at by induction, and a correction for the sum of the higher terms thus concluded. This correction must always be regarded as uncertain by a not inconsiderable fraction of its amount, and several of the results are therefore quite uncertain.

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{d\,k_\epsilon}{d\,x}$

Terms independent of m	+ .500 000
multiplied by m^2	— .006 492
m^3	— .000 050
m^4	+ .000 602
m^5	+ .000 246
m^6	+ .000 063
m^7	+ .000 012
Higher terms by induction	2
Sum	+0.494 383

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{e}\,\frac{d\,k_\epsilon}{d\,e}$

Terms multiplied by m^2	—.0042
m^3	—.0058
m^4	—.0028
Higher terms by induction	—.0010
Sum	—.0138

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{\gamma}\,\frac{d\,k_\epsilon}{d\,\gamma}$

Terms multiplied by m^2	+.0168
m^3	—.0009
m^4	—.0008
Higher terms by induction	—.0004
Sum	+.0147

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{d\,k_\epsilon}{d\,x'}$

Terms multiplied by m^2	+.005 565
m^3	+.000 045
m^4	—.000 556
m^5	—.000 231
m^6	—.000 059
m^7	—.000 012
Higher terms by induction	—.000 003
Sum	+.004 749

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{e'}\,\frac{d\,k_\epsilon}{d\,e'}$

Terms multiplied by m^2	—.00 560
m^4	+.00 131
m^5	+.00 085
Higher terms by induction	+.00 044
Sum	—.00 300

From the magnitude of the term concluded from induction we may conclude that this quantity is uncertain by $\frac{1}{20}$ of its entire amount, in consequence of the slowness with which the series converges.

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{e^2}\,\frac{d\,k_\pi}{d\,x}$

Terms independent of m	—.2502
multiplied by m^2	+.0590
m^3	+.0320
m^4	+.0111
Higher terms by induction	+.0030
Sum	—.1451

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{e}\,\frac{d\,k_\pi}{d\,e}$

Terms independent of m	— 1.0016
multiplied by m^2	+ .0335
m^3	+ .0129
m^4	+ .0034
Higher terms by induction	+ .0008
Sum	— .9510

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{e^2\,\gamma}\,\frac{d\,k_\pi}{d\,\gamma}$

Terms independent of m	$+.003$
multiplied by m^2	$-.119$
Sum	$-.116$

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{e^2}\,\frac{d\,k_\pi}{d\,x'}$

Terms multiplied by m^2	$-.0506$
m^3	$-.0290$
m^4	$-.0102$
Higher terms by induction	— 0034
Sum	$-.0932$

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{e^2\,e'}\,\frac{d\,k_\pi}{d\,e'}$

Terms in m^2	-0.027

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{\gamma^2}\,\frac{d\,k_\theta}{d\,x}$

Terms independent of m	$-$ .99 852
multiplied by m^2	$-$ .01 128
m^3	$+$ .00 245
m^4	$+$ .00 035
Sum	$-$ 1.00 700

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{e\,\gamma^2}\,\frac{d\,k_\theta}{d\,e}$

Terms independent of m	$+1.98$
multiplied by m^2	-0.12
Sum	$+1.86$

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{\gamma}\,\frac{d\,k_\theta}{d\,\gamma}$

Terms independent of m	-3.9942
multiplied by m^2	$-$ .0064
m^3	$+$ .0010
m^4	$+$ .0001
Sum	-3.9995

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{\gamma^2}\,\frac{d\,k_\theta}{d\,w'}$

Terms multiplied by m^2	$-.00\,968$
m^3	$+.00\,220$
m^4	$+.00\,033$
Sum	$-.00\,715$

We therefore have the following numerical values of the derivatives, in which, however, the last figure is nearly always to be regarded as quite uncertain.

$$\frac{d\,k_\epsilon}{d\,w} = +.494\,383\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\epsilon}{d\,e} = -.0138\,e\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\epsilon}{d\gamma} = +.0147\,\gamma\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\epsilon}{d w'} = +.004\,749\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\epsilon}{d\,e'} = -.00\,300\,e'\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\pi}{d\,w} = -.000\,437\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\pi}{d\,e} = -.9510\,e\,\frac{m_2\,m_3}{a\,n} \tag{95}$$

$$\frac{d\,k_\pi}{d\,\gamma} = -.000\,35\,\gamma\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\pi}{d\,w'} = -.000\,281\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\pi}{d\,e'} = -.000\,08\,e'\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\theta}{d\,w} = -.002\,028\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\theta}{d\,e} = +.003\,75\,e\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\theta}{d\,\gamma} = -3.9995\,\gamma\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\theta}{d\,x'}=-.000\,014\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\theta}{d\,e'}=-.000\,032\,e'\,\frac{m_2\,m_3}{a\,n} \qquad (95)$$

The values of $\frac{d\,x}{d\,k_\epsilon}$, etc., may be obtained either by solving the equations (89), when they will be the coefficients of $\frac{d\,\mathrm{R}}{d\,\epsilon}$, etc., or by solving the equations (91). We thus find

$$\left(\frac{d\,x}{d\,k_\epsilon}\right)=2.0225\,\frac{a\,n}{m_2\,m_3}$$

$$\left(\frac{d\,x}{d\,k_\pi}\right)=-.0293\,\frac{a\,n}{m_2\,m_3}$$

$$\left(\frac{d\,x}{d\,k_\theta}\right)=+.0075\,\frac{a\,n}{m_2\,m_3}$$

$$\left(\frac{d\,e}{d\,k_\epsilon}\right)=-.000\,928\,\frac{a\,n}{e\,m_2\,m_3}$$

$$\left(\frac{d\,e}{d\,k_\pi}\right)=-1.0514\,\frac{a\,n}{e\,m_2\,m_3}$$

$$\left(\frac{d\,e}{d\,k_\theta}\right)=+.000\,09\,\frac{a\,n}{e\,m_2\,m_3}$$

$$\left(\frac{d\,\gamma}{d\,k_\epsilon}\right)=-.001\,045\,\frac{a\,n}{\gamma\,m_2\,m_3}$$

$$\left(\frac{d\,\gamma}{d\,k_\pi}\right)=-.000\,97\,\frac{a\,n}{\gamma\,m_2\,m_3}$$

$$\left(\frac{d\,\gamma}{d\,k_\theta}\right)=-0.250\,04\,\frac{a\,n}{\gamma\,m_2\,m_3}$$

Now, resuming the equations (90), let us begin with those terms $\frac{d\,x}{d\,t}$, etc., which are dependent only on the variations of the solar elements, and which are given in (93). We have put

$$\frac{d\,k_\epsilon}{d\,t'}=\frac{d\,k_\epsilon}{d\,x'}\frac{d\,x'}{d\,t}+\frac{d\,k_\epsilon}{d\,e'}\frac{d\,e'}{d\,t}$$

$$\frac{d\,k_\pi}{d\,t'}=\frac{d\,k_\pi}{d\,x'}\frac{d\,x'}{d\,t}+\frac{d\,k_\pi}{d\,e'}\frac{d\,e'}{d\,t}$$

$$\frac{d\,k_\theta}{d\,t'}=\frac{d\,k_\theta}{d\,x'}\frac{d\,x'}{d\,t}+\frac{d\,k_\theta}{d\,e'}\frac{d\,e'}{d\,t}$$

If we here substitute the numerical values of $\frac{d\,k}{d\,v'}$, etc., just found, and then substitute in (93) both these values and the values of $\frac{d\,v}{d\,k}$, etc., in (96), we have

$$\frac{d\,v}{d\,t} = -.0096\frac{d\,v'}{d\,t} + .006\,07\,e'\frac{d\,e'}{d\,t}$$

$$e\frac{d\,e}{d\,t} = -.000\,29\frac{d\,v'}{d\,t} - .000\,09\,e'\frac{d\,e'}{d\,t}$$

$$\gamma\frac{d\,\gamma}{d\,t} = .000\,001\,\frac{d\,v'}{d\,t} - .000\,011\,e'\frac{d\,e'}{d\,t}$$

and, by integration,

$$\begin{aligned} \delta\,v &= -.0096\;\delta\,v' + .003\,04\;\delta.e'^2 \\ \delta.e^2 &= -.000\,58\;\delta\,v' - .000\,09\;\delta.e'^2 \\ \delta.\gamma^2 &= \;\;.\;\;.\;\;.\;\;.\;\; -.000\,011\;\delta.e'^2 \end{aligned} \tag{96}$$

These quantities are to be substituted in $n\,\pi_1$, and ϑ_1, to obtain the corresponding variations of ε, π, and θ. We have first

$$\begin{aligned} \delta\,n &= -\frac{3}{2}n\,\delta\,v \\ &= +\frac{3}{2}n\left\{.0096\;\delta\,v' - .003\,04\;\delta.e'^2\right\} \end{aligned}$$

Taking the partial derivatives of π_1 and θ_1 (p. 190) with respect to v, v', and e' we find the following terms of

$$\frac{1}{n}\frac{d\,\pi_1}{d\,v}$$

Terms in m^2	.00 619
m^3	873
m^4	443
m^5	179
m^6	68
m^7	25
m^8	10
m^9	4
Higher terms by induction	2
Sum	.02 223

Formation of $\frac{1}{n}\frac{d\pi_1}{dw'}$

Terms in m_2	—.0123
m_3	—.0131
m_4	—.0059
m_5	—.0022
m_6	—.0008
m_7	—.0003
m_8 and upwards	—.0002
Sum	—.0348

Formation of $\frac{1}{n}\frac{2}{e'}\frac{d\pi_1}{de'}$

Terms in m_2	+.0063
m_3	+.0108
m_4	+.0075
m_5	+.0040
Higher terms by induction	+.0032
Sum	+.0318

The derivatives of θ_1 are much more convergent. We find

$$\frac{1}{n}\frac{d\theta_1}{dw} = -.005\,61$$

$$\frac{1}{n}\frac{d\theta_1}{dw'} = +.011\,64$$

$$\frac{1}{n}\frac{2}{e'}\frac{d\theta_1}{de'} = -.005\,37$$

The terms of ε, π, and θ, which are independent of the derivatives of R with respect to the lunar elements, are, by (92)

$$\begin{aligned} \delta\varepsilon &= \int \delta n\,dt \\ \delta\pi &= \int \delta\pi_1 dt \qquad (97) \\ \delta\theta &= \int \delta\theta_1 dt \end{aligned}$$

We have, from what precedes

$$\delta n = -\frac{3}{2} n \, \delta v$$

$$\delta \pi_1 = n \left\{ +.022\,23\, \delta v - .0348\, \delta v' + .0318\, \delta .e'^2 \right\}$$

$$\delta \theta_1 = n \left\{ -.005\,61\, \delta v + .011\,64\, \delta v' - .005\,37\, \delta .e'^2 \right\}$$

Substituting for $\delta \alpha$ its value in $\delta v'$ and $\delta .e'^2$ just found, these values become

$$\begin{aligned} \delta n &= n \{ .0144\, \delta v' - .004\,56\, \delta .e'^2 \} \\ \delta \pi_1 &= n \{ -.0350\, \delta v' + .0319\, \delta .e'^2 \} \\ \delta \theta_1 &= n \{ +.01\,169\, \delta v' - .005\,39\, \delta .e'^2 \} \end{aligned} \tag{98}$$

The forms of the integrals deduced from these values will be different according as we consider the secular or periodic terms. There are, however, no secular terms in $\delta v'$. Corresponding to the adopted values of the planetary masses are the following secular perturbations of e'

$$\delta .e' = -8''.73\, T - 0''.028\, T^2$$

T being the number of centuries from 1800. From this and the adopted constant term of e' we derive

$$\delta .e'^2 = -0''.2926\, T - 0''.000\,56\, T^2$$

and hence

$$\begin{aligned} \delta \epsilon &= 5''.60\, T^2 + 0''.007\, T^3 \\ \delta \pi &= -39''.3\, T^2 - 0''.050\, T^3 \\ \delta \theta &= 6''.64\, T^2 + 0''.008\, T^3 \end{aligned} \tag{99}$$

From the manner in which these quantities are obtained it may be shown that the coefficient of T^2 in $\delta \epsilon$ depends mainly on the value of $\frac{d k_\epsilon}{d e'}$, and the value of $\delta \pi$ mainly on that of $\frac{d \pi_1}{d e'}$. We may, therefore, conclude from the defect of convergence in the series which express these quantities that the first quantity, the secular acceleration of the mean longitude, is uncertain by more than $\frac{1}{20}$, and the second by $\frac{1}{50}$ of its entire amount, while the third is nearly exact. Although all the observations hitherto made on the Moon would not enable us to detect errors of this magnitude, a more accurate determination would be desirable from a theoretical point of view. But this can not be made without further investigations into the problem of three bodies, which it is not the object of the present paper to undertake.

Let us now take up the periodic terms of $\delta x'$ and $\delta e'$. If we put

$$\delta x' = a_c \cos \mathrm{N} + a_s \sin \mathrm{N}$$

$$\delta e' = e_c \cos \mathrm{N} + e_s \sin \mathrm{N}$$

where N is an angle increasing uniformly with the time and therefore of the form $\mathrm{N}_0 + bt$, and if we substitute these values in the equations for δn, $\delta \pi_1$, and $\delta \theta_1$, and integrate, we find

$$\begin{aligned}\delta \varepsilon = +\ & .0144\, \nu \{a_c \sin \mathrm{N} - a_s \cos \mathrm{N}\} \\ & -.000\,155\, \nu \{e_c \sin \mathrm{N} - e_s \cos \mathrm{N}\}\end{aligned}$$

or,

$$\delta \varepsilon = (.0144\, a_c - .000\,155\, e_c)\, \nu \sin \mathrm{N} - (.0144\, a_s - .000\,155\, e_s)\, \nu \cos \mathrm{N} \qquad (100)$$

where we put

$$\nu = \frac{n}{b}$$

It will be seen that ν represents the quotient of the period of the inequality divided by the period of revolution of the Moon.

Applying the same process to π and θ, we have

$$\left.\begin{aligned}\delta \pi = (-.0350\, a_c + .00\,106\, e_c)\, \nu \sin \mathrm{N} & \\ + (0350\, a_s - .00\,106\, e_s)\, \nu \cos \mathrm{N} & \\ \delta \theta = (.0117\, a_c - .00\,018\, e_c)\, \nu \sin \mathrm{N} + (-\ 0117\, a_s + 00\,018\, e_s)\, \nu \cos \mathrm{N} &\end{aligned}\right\} \qquad (101)$$

The derivation of the numerical results of these formulæ requires the determination of the periodic inequalities of the solar elements produced by the action of each planet. The results will be found in Chapter III.

§ 13.

REMARKS ON THE RELATION OF THE PRECEDING THEORY TO THAT OF DELAUNAY.

The relation of the preceding theory to that of Delaunay is so remarkable as to deserve special attention. Delaunay continually reduced the differential equations to the canonical form

$$\frac{d\mathrm{L}}{dt} = \frac{d\mathrm{R}}{dl} \qquad \frac{d\mathrm{L}}{dt} = -\frac{d\mathrm{R}}{d\mathrm{L}}$$

$$\frac{d\mathrm{G}}{dt} = \frac{d\mathrm{R}}{dg} \qquad \frac{dg}{dt} = -\frac{d\mathrm{R}}{d\mathrm{G}}$$

$$\frac{d\mathrm{H}}{dt} = \frac{d\mathrm{R}}{dh} \qquad \frac{dh}{dt} = -\frac{d\mathrm{R}}{d\mathrm{H}}$$

where l, g, and h correspond as follows to the variable angles in the theory in the present paper:

$$l = \varepsilon - \pi$$

$$g = \pi - \theta$$

$$h = \theta$$

Here R may be regarded as the potential of any disturbing force whatever tending to change the lunar elements. Comparing DELAUNAY's R with the equations (14) of the present paper, and the preceding equations with (65), it will be seen that those parts of our k_ε, k_π, and k_θ which are derived from the coordinates of the Moon alone, which parts are the only ones yet considered, should be related to the constants L, G, H, of DELAUNAY in the following way:

$$k_\varepsilon = \mu_2 \mathrm{L}$$

$$k_\pi = \mu_2 (\mathrm{G} - \mathrm{L})$$

$$k_\theta = \mu_2 (\mathrm{H} - \mathrm{G})$$

The final values of these constants, in terms of a, e, and γ, are given by DELAUNAY in Vol. II of his work, pp. 235–236. We could, therefore, immediately compare these expressions with the values of our k_ε, etc., finally obtained, but for the circumstance that the elements a, e, and γ here used are not identical with those in which the coordinates of the Moon are finally expressed at the end of the work. To reduce the one to the other, certain substitutions must be made, which are given on page 800 The identity of the expressions which would be derived by making these substitutions in L, G, H, with the values which I have derived from the Moon's coordinates would afford so remarkable confirmation of the numerical correctness of DELAUNAY's results, that I have deemed it worth while to make the transformation in question. The results are as follows:

$$\mathrm{L} = \sqrt{a\,\mu}\left\{\begin{array}{l} 1 + \left(-\frac{1}{3} + \frac{3}{2}\gamma^2 - \gamma^4 - \frac{1}{2}e'^2 - \frac{5}{8}e'^4 + \frac{9}{4}\gamma^2 e'^2 - \frac{3}{8}e^2 + \frac{5}{4}\gamma^2 e^2 - \frac{9}{16}e'^2 e^2 \right. \\ \qquad \left. - \frac{1}{32}e^4\right) m^2 \\ \quad + \left(-\frac{225}{32}e^2 - \frac{9}{8}\gamma^2 - \frac{825}{32}e'^2 e^2\right) m^3 + \left(+\frac{895}{288} - \frac{1585}{128}\gamma^2 + \frac{3995}{192}e'^2 \right. \\ \qquad \left. - \frac{23\,255}{512}e^2\right) m^4 \\ \quad + \left(+\frac{1261}{96} + \frac{11\,349}{64}e'^2\right) m^5 + \frac{3028}{81}m^6 + \frac{76\,741}{864}m^7 \end{array}\right\}$$

$$G=\sqrt{a\mu}\left\{\begin{aligned}&1-\frac{1}{2}e^2+\frac{5}{8}\gamma^4e^2-\frac{1}{8}e^4-\frac{5}{16}\gamma^2e^4-\frac{1}{16}e^6\\&+\Big(-\frac{1}{3}+\frac{3}{2}\gamma^2-\gamma^4-\frac{1}{2}e'^2-\frac{5}{8}e'^4+\frac{9}{4}\gamma^2e'^2+\frac{1027}{384}e^2-\frac{75}{8}\gamma^2e^2\\&\qquad-\frac{381}{128}e'^2e^2-\frac{2081}{384}e^4\Big)m^2\\&+\Big(+\frac{2145}{256}e^2-\frac{9}{8}\gamma^2-\frac{25\,495}{512}e'^2e^2\Big)m^3+\Big(+\frac{895}{288}-\frac{1585}{128}\gamma^2\\&\qquad+\frac{3995}{192}e'^2+\frac{327\,263}{36\,864}e^2\Big)m^4\\&+\Big(+\frac{1261}{96}+\frac{11\,349}{64}e'^2\Big)m^5+\frac{3028}{81}m^6+\frac{76\,741}{864}m^7\end{aligned}\right\}$$

$$H=\sqrt{a\mu}\left\{\begin{aligned}&1-2\gamma^2-\frac{1}{2}e^2+\gamma^2e^2-\frac{65}{8}\gamma^4e^2-\frac{1}{8}e^4+\frac{33}{32}\gamma^2e^4-\frac{1}{16}e^6\\&+\Big(-\frac{1}{3}+\frac{91}{96}\gamma^2+\frac{111}{32}\gamma^4-\frac{1}{2}e'^2-\frac{5}{8}e'^4+\frac{1027}{384}e^2-\frac{3901}{192}\gamma^2e^2\\&\qquad-\frac{381}{128}e'^2e^2-\frac{2081}{384}e^4+\frac{27}{32}e'^2\gamma^2\Big)m^2\\&+\Big(+\frac{2145}{256}e^2+\frac{3}{64}\gamma^2-\frac{25\,495}{512}e'^2e^2\Big)m^3+\Big(+\frac{895}{288}-\frac{98\,113}{9216}\gamma^2\\&\qquad+\frac{3995}{192}e'^2+\frac{327\,263}{36\,864}e^2\Big)m^4\\&+\Big(+\frac{1261}{96}+\frac{11\,349}{64}e'^2\Big)m^5+\frac{3028}{81}m^6+\frac{76\,741}{864}m^7\end{aligned}\right\}$$

The above relation affords a remarkable method of determining the secular variations of the Moon's mean motion, and of the motions of its perigee and node. From the equations (94) of the preceding section it appears that these variations are immediately obtainable from the three equations

$$\frac{d\,L}{d\,a}\frac{d\,a}{d\,t}+\frac{d\,L}{d\,e}\frac{d\,e}{d\,t}+\frac{d\,L}{d\,\gamma}\frac{d\,\gamma}{d\,t}+\frac{d\,L}{d\,e'}\frac{d\,e'}{d\,t}=0$$

$$\frac{d\,G}{d\,a}\frac{d\,a}{d\,t}+\frac{d\,G}{d\,e}\frac{d\,e}{d\,t}+\frac{d\,G}{d\,\gamma}\frac{d\,\gamma}{d\,t}+\frac{d\,G}{d\,e'}\frac{d\,e'}{d\,t}=0$$

$$\frac{d\,H}{d\,a}\frac{d\,a}{d\,t}+\frac{d\,H}{d\,e}\frac{d\,e}{d\,t}+\frac{d\,H}{d\,\gamma}\frac{d\,\gamma}{d\,t}+\frac{d\,H}{d\,e'}\frac{d\,e'}{d\,t}=0$$

The value of the secular variation of e' is to be inserted in the last terms, and the equations are to be solved for

$$\frac{da}{dt},\ \frac{de}{dt},\ \text{and}\ \frac{d\gamma}{dt}$$

We then have

$$\frac{d^2\varepsilon}{dt^2}=\frac{dn}{da}\frac{da}{dt}$$

$$\frac{d^2\pi}{dt^2}=\frac{d\pi_1}{da}\frac{da}{dt}+\frac{d\pi_1}{de}\frac{de}{dt}+\frac{d\pi_1}{d\gamma}\frac{d\gamma}{dt}+\frac{d\pi_1}{de'}\frac{de'}{dt}$$

$$\frac{d^2\theta}{dt^2}=\frac{d\theta_1}{da}\frac{da}{dt}+\frac{d\theta_1}{de}\frac{de}{dt}+\frac{d\theta_1}{d\gamma}\frac{d\gamma}{dt}+\frac{d\theta_1}{de'}\frac{de'}{dt}$$

π_1 and θ_1 being the secular motions of π and θ, as given in the preceding section p. 190.

The analytic expression of these variations in this way offers no difficulty as far as the terms of the seventh order. Above this, however, we meet with the difficulty that the transformations given by DELAUNAY, on page 800, do not extend in any case beyond terms of this order in e, γ, and m. Moreover, DELAUNAY'S supplementary expression for R, on page 742, does not seem to afford the means of carrying the transformation further. Were it required to write the value of L, or of k_e, with a higher degree of precision than this, it might be possible to do so inductively from DELAUNAY'S expression for the secular variation of the Moon's mean motion given in the *Comptes Rendus*, Vol. 72, p. 496, which extends to terms in m^9. This course does not, however, seem necessary, as the coefficient in question, when carried to the fifth power of m, is sufficiently accurate for all purposes except the computation of the secular acceleration itself.

It might be remarked, in this connection, that DELAUNAY never published any details of the process by which he effected the transformation of the values of e, γ, and a, beyond terms of the seventh order in m. It is understood that these were reserved for his intended third volume, which was never prepared for the press.

§ 14.

TRANSFORMATION OF THE PERTURBATIVE FUNCTION R AND OF ITS DERIVATIVES.

We have in the first chapter reduced the function R to the following form:

$$R=m_4\,\mu_1\left(\frac{1}{\rho}+\frac{x'\,x_4+y'\,y_4+z'\,z_4}{r_4^3}\right)$$

$$+\frac{1}{2}m_4\,\mu_2\left(\frac{3\,\Delta^2}{\rho^5}-\frac{r^2}{\rho^3}\right)$$

where

$$\Delta=x(x_4+x')+y(y_4+y')+z(z_4+z')$$

In this form, however, the function R is subject to a serious inconvenience, in containing the square of Δ, which is a function of the elements of the Sun, Moon, and planet. This makes it very difficult to select all the terms corresponding to any particular combination of the varying angles. We shall, therefore, express R in the form of a series of products, in which that portion of its derivatives which contains the disturbed elements of the Sun and Moon shall be separated from that portion which depends upon the elements of the planet.

We have from the preceding expression for Δ

$$\begin{aligned}\Delta^2=&(x'+x_4)^2x^2+(y'+y_4)^2y^2+(z'+z_4)^2z^2\\&+2(x'+x_4)\ (y'+y_4)\,xy\\&+2(y'+y_4)\ (z'+z_4)\,yz\\&+2(z'+z_4)\ (x'+x_4)\,zx\end{aligned}$$

We have now by differentiating the preceding expression for R with respect to any element a

$$\begin{aligned}\frac{dR}{da}=m_4\mu_1\Big\{&\Big(\frac{x_4}{r_4^3}-\frac{x'}{\rho^3}-\frac{x_4}{\rho^3}\Big)\frac{dx'}{da}\\&+\Big(\frac{y_4}{r_4^3}-\frac{y'}{\rho^3}-\frac{y_4}{\rho^3}\Big)\frac{dy'}{da}\\&+\Big(\frac{z_4}{r_4^3}-\frac{z'}{\rho^3}-\frac{z_4}{\rho^3}\Big)\frac{dz'}{da}\Big\}\\+m_4\mu_2\Big\{&\Big(3\frac{(x'+x_4)^2}{\rho^5}-\frac{1}{\rho^3}\Big)x\frac{dx}{da}\\&+\Big(3\frac{(y'+y_4)^2}{\rho^5}-\frac{1}{\rho^3}\Big)y\frac{dy}{da}\\&+\Big(3\frac{(z'+z_4)^2}{\rho^5}-\frac{1}{\rho^3}\Big)z\frac{dz}{da}\\&+\frac{3(x'+x_4)\ (y'+y_4)}{\rho^5}\Big(x\frac{dy}{da}+y\frac{dx}{da}\Big)\\&+\frac{3(y'+y_4)\ (z'+z_4)}{\rho^5}\Big(y\frac{dz}{da}+z\frac{dy}{da}\Big)\\&+3\frac{(z'+z_4)\ (x'+x_4)}{\rho^5}\Big(z\frac{dx}{da}+x\frac{dz}{da}\Big)\Big\}\end{aligned}\qquad(102)$$

Putting the latter part of this expression in this form affords us a great advantage in the selection of the terms which may become sensible. We see that in each of the last six products the first factor contains only the coordinates of the Sun and planet, and not those of the Moon. Therefore, when we omit, as we must in a first approximation, the squares and products of the masses of the planets, the factors in question admit of being developed as a function of the mean anomalies of the Sun and planet. Each term of this development is of the form

$$K\,{}^{\sin}_{\cos}\,(i\,g'+j\,g_4)=K\,{}^{\sin}_{\cos}\,U$$

g' being the mean anomaly of the Sun, g_4 that of the planet, and K a function of the eccentricities, mean distances, and inclinations. As it is necessary to carry the development to high multiples of i and j, the terms, in the case of a single planet, may amount to several hundred in number.

The second factor of each product is a function only of the coordinates of the Moon relatively to the Earth, which we have represented by x, y, z, or, rather, it is a function of the elements of the motion of the Moon around the Earth. But these necessarily include the elements of the Sun, multiplied by its disturbing force. As we have developed x, y, and z, each term of the second factor will be of the form

$$\varkappa\,{}^{\sin}_{\cos}\,(i_1\,l+i_2\,\pi+i_3\,\theta+i_4\,l'+i_5\,\pi'+i_6\,\theta')$$

In the first approximation we may regard π' and θ', or the solar node and perigee as constant, and the terms in question may be put in the form

$$\varkappa\,{}^{\sin}_{\cos}\,(i_1\,l+i_2\,\pi+i_3\,\theta+i_4\,g')=\varkappa\,{}^{\sin}_{\cos}\,N$$

Although we have here four variable angles, all of whose multiples are to be included, yet they converge so rapidly that the terms we have retained in the lunar coordinates include all that can be expected to give any appreciable result.

Having thus developed each factor, if we form their products, we shall find each term of $\frac{dR}{da}$, which contains the lunar elements, to be of the form

$$\frac{1}{2}m_4\,\mu_2\,K\,{}^{\sin}_{\cos}\,(N\pm U)$$

The derivatives of each of the elements contain these terms multiplied by a factor which is constant in the approximation. If, then, we represent by s the coefficient of the time in $N\pm n$, we see that the integrated values of all the elements of the Moon contain terms of the form

$$\frac{c}{s}\,{}^{\sin}_{\cos}\,(N\pm U)$$

and ε, π, and θ contain in addition terms of the form

$$\frac{c'}{s^2}\,{}^{\sin}_{\cos}\,(N\pm U)$$

We hence derive the two following theorems of the motion of the Moon disturbed by a planet:

1. *The elements of the orbit of the Moon disturbed by a planet contain an infinity of terms of the form*

$$\mathrm{c}\,\frac{\sin}{\cos}\,(i_1\,l+i_2\,\pi+i_3\,\theta+i_4\,g'+i_5\,g_4)$$

2. *If any such term is of very long period it may become sensible owing to the smallness of the divisor, the first and second powers of which enter into the coefficient* c.

§ 15.

FORM OF THE PERTURBATIONS ARISING FROM THE SECOND TERM OF R.

Let us take up the second part of the expression (102) which we shall call $\frac{d\,\mathrm{R}_2}{d\,a}$, and examine more fully the perturbations to which it may give rise. We have just shown that the coefficients which depend on the coordinates of the Sun and planetmay be developed in multiples of their mean anomalies. Let us then put

$$\begin{aligned}
&\frac{3}{2}\frac{(x'+x_4)^2}{\rho^5}-\frac{1}{2}\frac{1}{\rho^3}=\mathrm{K}_1\cos\mathrm{U}+\mathrm{K}'_1\sin\mathrm{U}\\
&\frac{3}{2}\frac{(y'+y_4)^2}{\rho^5}-\frac{1}{2}\frac{1}{\rho^3}=\mathrm{K}_2\cos\mathrm{U}+\mathrm{K}'_2\sin\mathrm{U}\\
&\frac{3}{2}\frac{(z'+z_4)^2}{\rho^5}-\frac{1}{2}\frac{1}{\rho^3}=\mathrm{K}_3\cos\mathrm{U}+\mathrm{K}'_3\sin\mathrm{U}\\
&\frac{3\,(x'+x_4)\,(y'+y_4)}{\rho^5}=\mathrm{K}_4\cos\mathrm{U}+\mathrm{K}'_4\sin\mathrm{U}\\
&\frac{3\,(y'+y_4)\,(z'+z_4)}{\rho^5}=\mathrm{K}_5\cos\mathrm{U}+\mathrm{K}'_5\sin\mathrm{U}\\
&\frac{3\,(z'+z_4)\,(x'+x_4)}{\rho^5}=\mathrm{K}_6\cos\mathrm{U}+\mathrm{K}'_6\sin\mathrm{U}
\end{aligned}\qquad(103)$$

Here K and K′ are numerical coefficients, calculated in the manner to be hereafter described, and U an angle of the form

$$i'\,g'+i_4\,g_4$$

Let us also put

$$\begin{aligned}
&\frac{x^2}{a^2}=\varkappa_1\cos\mathrm{N}\\
&\frac{y^2}{a^2}=\varkappa_2\cos\mathrm{N}
\end{aligned}\qquad(104)$$

$$\frac{z^2}{a^2}=\kappa_3 \cos \mathrm{N}$$

$$\frac{x\,y}{a^2}=\kappa_4 \sin \mathrm{N}$$

$$\frac{y\,z}{a^2}=\kappa_5 \sin \mathrm{N} \qquad (104)$$

$$\frac{z\,x}{a^2}=\kappa_6 \sin \mathrm{N}$$

Each N is of the form

$$i\,l+i'\,\pi+i''\,\theta+i'''g'$$

and each *K* a function of *a*, *e*, *γ*, *a′*, *e′* and *γ′*.

We therefore have for the derivatives relatively to *ε*,

$$2x\frac{d\,x}{d\,\varepsilon}=-i\,a^2\kappa_1 \sin \mathrm{N}$$

$$2y\frac{d\,y}{d\,\varepsilon}=-i\,a^2\kappa_2 \sin \mathrm{N}$$

$$2z\frac{d\,z}{d\,\varepsilon}=-i\,a^2\kappa_3 \sin \mathrm{N} \qquad (105)$$

$$x\frac{d\,y}{d\,\varepsilon}+y\frac{d\,x}{d\,\varepsilon}=i\,a^2\kappa_4 \cos \mathrm{N}$$

$$y\frac{d\,z}{d\,\varepsilon}+z\frac{d\,y}{d\,\varepsilon}=i\,a^2\kappa_5 \cos \mathrm{N}$$

$$z\frac{d\,x}{d\,\varepsilon}+x\frac{d\,z}{d\,\varepsilon}=i\,a^2\kappa_6 \cos \mathrm{N}$$

while the derivatives with respect to *π* and *θ* are of the same form, except that *i′* and *i″*, respectively, are to be substituted for *i*. For the derivatives relatively to *ϖ*, *e*, and *γ*, we have

$$2x\frac{d\,x}{d\varpi}=a^2\left(2\,\kappa_1+\frac{d\,\kappa_1}{d\,\varpi}\right) \cos \mathrm{N}$$

$$2\,y\frac{d\,y}{d\varpi}=a^2\left(2\,\kappa_2+\frac{d\,\kappa_2}{d\,\varpi}\right) \cos \mathrm{N} \qquad (106)$$

$$2\,z\frac{d\,z}{d\varpi}=a^2\left(2\,\kappa_3+\frac{d\,\kappa_3}{d\varpi}\right) \cos \mathrm{N}$$

$$x\frac{d\,y}{d\,\varpi}+y\frac{d\,x}{d\,\varpi}=a^2\left(2\,\varkappa_4+\frac{d\,\varkappa_4}{d\,\varpi}\right)\sin \mathrm{N}$$

etc., etc., etc.

$$2\,x\frac{d\,x}{d\,e}=a^2\frac{d\,\varkappa_1}{d\,e}\cos \mathrm{N}$$

. . . .

$$x\frac{d\,y}{d\,e}+y\frac{d\,x}{d\,e}=a^2\frac{d\,\varkappa_4}{d\,e}\sin \mathrm{N}$$

etc., etc.,

while the derivatives with respect to γ are of the same form.

Substituting in the general form (102) the expressions in (103), (105) and (106) we find for the result of any combination of terms

$$\frac{d\,\mathrm{R}_2}{d\,\varepsilon}=\frac{1}{2}\,i m_4\mu_2(\pm K'_1\varkappa_1\pm K'_2\varkappa_2\pm K'_3\varkappa_3+K_4\varkappa_4+K_5\varkappa_5+K_6\varkappa_6)\,a^2\cos(\mathrm{N}\pm\mathrm{U})$$
$$+\frac{1}{2}\,i m_4\mu_2(-K_1\varkappa_1-K_2\varkappa_2-K_3\varkappa_3\pm K'_4\varkappa_4\pm K'_5\varkappa_4\pm K'_6\varkappa_6)\,a^2\sin(\mathrm{N}\pm\mathrm{U})$$

$$\frac{d\,\mathrm{R}_2}{d\,\varpi}=\frac{1}{2}\,m_4\mu_2\Big(2\,K_1\varkappa_1+\;\ldots\;+K_1\frac{d\,\varkappa_1}{d\,\varpi}+K_2\frac{d\,\varkappa_2}{d\,\varpi}+K_3\frac{d\,\varkappa_3}{d\,\varpi}\mp K'_4\frac{d\,\varkappa_4}{d\,\varpi}\mp K'_5\frac{d\,\varkappa_5}{d\,\varpi}$$
$$\mp K'_6\frac{d\,\varkappa_6}{d\,\varpi}\Big)\,a^2\cos(\mathrm{N}\pm\mathrm{U})$$

$$+\frac{1}{2}\,m_4\mu_2\Big(\pm 2\,K'_1\varkappa_1\pm\;\ldots\;\pm K'_1\frac{d\,\varkappa_1}{d\,\varpi}\pm K'_2\frac{d\,\varkappa_2}{d\,\varpi}\pm K'_3\frac{d\,\varkappa_3}{d\,\varpi}+K^4\frac{d\,\varkappa_4}{d\,\varpi}$$
$$+K_5\frac{d\,\varkappa_5}{d\,\varpi}+K_6\frac{d\,\varkappa_6}{d\,\varpi}\Big)\,a^2\sin(\mathrm{N}\pm\mathrm{U})$$

If we put for brevity

$$\begin{aligned}K&=\pm K'_1\varkappa_1\pm K'_2\varkappa_2\pm K'_3\varkappa_3+K_4\varkappa_4+K_5\varkappa_5+K_6\varkappa_6\\ K'&=\quad K_1\varkappa_1+K_2\varkappa_2+K_3\varkappa_3\mp K'_4\varkappa_4\mp K'_5\varkappa_5\mp K'_6\varkappa_6\end{aligned}\qquad(107)$$

we shall have

$$\begin{aligned}\frac{d\,\mathrm{R}_2}{d\,\varepsilon}&=\frac{1}{2}\,i m_4\mu_2a^2\Big\{K\cos(\mathrm{N}\pm\mathrm{U})-K'\sin(\mathrm{N}\pm\mathrm{U})\Big\}\\ \frac{d\,\mathrm{R}_2}{d\,\pi}&=\frac{1}{2}\,i' m_4\mu_2a^2\Big\{K\cos(\mathrm{N}\pm\mathrm{U})-K'\sin(\mathrm{N}\pm\mathrm{U})\Big\}\\ \frac{d\,\mathrm{R}_2}{d\,\theta}&=\frac{1}{2}\,i'' m_4\mu_2a^2\Big\{K\cos(\mathrm{N}\pm\mathrm{U})-K'\sin(\mathrm{N}\pm\mathrm{U})\Big\}\end{aligned}\qquad(108)$$

$$\frac{d\,\mathrm{R}_2}{d\,\varpi}=\frac{1}{2}\,m_4\mu_2a^2\Big\{\left(2\,K+\frac{d\,K}{d\,\varpi}\right)\sin(\mathrm{N}\pm\mathrm{U})+\left(2\,K'+\frac{d\,K_1}{d\,\varpi}\right)\cos(\mathrm{N}\pm\mathrm{U})\Big\}$$

$$\frac{d\,\mathrm{R}_2}{d\,e}=\frac{1}{2}m_4\mu_2a^2\left\{\frac{d\,K}{d\,e}\sin(\mathrm{N}\pm\mathrm{U})+\frac{d\,K'}{d\,e}\cos(\mathrm{N}\pm\mathrm{U})\right\}$$

$$\frac{d\,\mathrm{R}_2}{d\,\gamma}=\frac{1}{2}m_4\mu_2a^2\left\{\frac{d\,K}{d\,\gamma}\sin(\mathrm{N}\pm\mathrm{U})+\frac{d\,K'}{d\,\gamma}\cos(\mathrm{N}\pm\mathrm{U})\right\}$$

These values of the derivatives of R being substituted in the equations (89) and (90), will give the values of the derivatives of the elements with respect to the time.

§ 16.

FINAL FORM OF THE DIFFERENTIAL EQUATIONS FOR THE VARIATIONS OF THE LUNAR ELEMENTS.

Returning once more to the equations (89) and (90), we have to find what terms are produced by the preceding value of R_2. If, in these equations, we omit the derivatives with respect to t' in the second members, the effect of which we have just computed, substitute for the derivatives of R those with respect to R_2 just found, and for the factors $\left(\frac{d\,w}{d\,k_4}\right)$, etc., their numerical values found in (96) the equations (89) become

$$\begin{aligned}\frac{d\,w}{d\,t}&=\mu_4a^3n\{\ 1.0112\,i-.0146\,i'+.0037\,i''\}\ \{\mathrm{K}\cos(\mathrm{N}\pm\mathrm{U})-\mathrm{K}'\sin(\mathrm{N}\pm\mathrm{U})\}\\ e\frac{d\,e}{d\,t}&=\mu_4a^3n\{-.0005\,i-.5257\,i'+.0000\,i''\}\ \{\mathrm{K}\cos(\mathrm{N}\pm\mathrm{U})-\mathrm{K}'\sin(\mathrm{N}\pm\mathrm{U})\}\\ \gamma\frac{d\,\gamma}{d\,t}&=\mu_4a^3n\{-.0005\,i-.0005\,i'-.1250\,i''\}\ \{\mathrm{K}\cos(\mathrm{N}\pm\mathrm{U})-\mathrm{K}'\sin(\mathrm{N}\pm\mathrm{U})\}\end{aligned}\qquad(109)$$

where we put, for brevity,

$$\mu_4=\frac{m_4}{m_2+m_3}$$

or the ratio of the mass of the planet to the sum of the masses of the Earth and Moon. Integrating these equations, and putting, as before, ν, the quotient of the Moon's mean motion by the coefficient of the time in the angle $\mathrm{N}\pm\mathrm{U}$, we have

$$\begin{aligned}\delta w&=\mu_4a^3\nu\{\ 1.0112\,i-.0146\,i'+.0037\,i''\}\ \{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U})+\mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}\\ e\,\delta e&=\mu_4a^3\nu\{-.0005\,i-.5257\,i'+.0000\,i''\}\ \{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U})+\mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}\\ \gamma\,\delta\gamma&=\mu_4a^3\nu\{-.0005\,i-.0005\,i'-.1250\,i''\}\ \{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U})+\mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}\end{aligned}\qquad(110)$$

There can be no terms corresponding to the case of $\mathrm{N}\pm\mathrm{U}=0$, because U contains neither ϵ, π, nor θ, and therefore in this case N can contain neither ϵ, π, nor θ, and therefore i, i', and i'' will all be zero, so that the coefficient of the angle will vanish.

These values of δw, δe, and $\delta \gamma$ are now to be substituted in the derivatives of n, π_1, and θ_1, and the latter are then to be integrated. If we put

$$\delta n = \frac{d n}{d w} \delta w$$

$$\delta \pi_1 = \frac{d \pi_1}{d w} \delta w + \frac{d \pi_1}{d e} \delta e + \frac{d \pi_1}{d \gamma} \delta \gamma$$

$$\delta \theta_1 = \frac{d \theta_1}{d w} \delta w + \frac{d \theta_1}{d e} \delta e + \frac{d \theta_1}{d \gamma} \delta \gamma$$

the values of $\delta \varepsilon$, $\delta \pi$, and $\delta \theta$ will be given by the equations

$$\delta \varepsilon = \int \delta n \, d t - \int \frac{d \mathrm{R}}{d k_\varepsilon} d t$$

$$\delta \pi = \int \delta \pi_1 \, d t - \int \frac{d \mathrm{R}}{d k_\pi} d t$$

$$\delta \theta = \int \delta \theta_1 \, d t - \int \frac{d \mathrm{R}}{d k_\theta} d t$$

Substituting the numerical values of $\frac{d n}{d w}$, $\frac{d \pi_1}{d w}$, etc., we have

$$\delta n = -\frac{3}{2} n \delta w$$

$$\delta \pi_1 = n \{ \quad .02223 \, \delta w - .0195 \, e \, \delta e - .0994 \, \gamma \, \delta \gamma \}$$

$$\delta \theta_1 = n \{ -.00561 \, \delta w - .0236 \, e \, \delta e + .0149 \, \gamma \, \delta \gamma \}$$

Substituing the preceding values of δw, $e \, \delta e$, and $\gamma \, \delta \gamma$, and integrating, we have

$$\delta \varepsilon = \mu_4 a^3 \nu^2 \{ -1.517 \, i + .022 \, i' - .005 \, i'' \} \{ -\mathrm{K} \cos (\mathrm{N} \pm \mathrm{U}) + \mathrm{K}' \sin (\mathrm{N} \pm \mathrm{U}) \}$$
$$- \int \frac{d \mathrm{R}}{d k_\varepsilon} d t$$

$$\delta \pi = \mu_4 a^3 \nu^2 \{ \quad .0225 \, i + .0099 \, i' + .0124 \, i'' \} \{ -\mathrm{K} \cos (\mathrm{N} \pm \mathrm{U}) \qquad (111)$$
$$+ \mathrm{K}' \sin (\mathrm{N} \pm \mathrm{U}) \} - \int \frac{d \mathrm{R}}{d k_\pi} d t$$

$$\delta \theta = \mu_4 a^3 \nu^2 \{ - \; .0057 \, i + .0124 \, i' - .0019 \, i'' \} \{ -\mathrm{K} \cos (\mathrm{N} \pm \mathrm{U}) + \mathrm{K}' \sin (\mathrm{N} \pm \mathrm{U}) \}$$
$$- \int \frac{d \mathrm{R}}{d k_\theta} d t$$

We are now to find the values of $\frac{d\,\mathrm{R}}{d\,k_\epsilon}$, etc., from the equations

$$\frac{d\,\mathrm{R}}{d\,k_\epsilon}=\left(\frac{d\,\alpha}{d\,k_\epsilon}\right)\frac{d\,\mathrm{R}}{d\,\varpi}+\left(\frac{d\,e}{d\,k_\epsilon}\right)\frac{d\,\mathrm{R}}{d\,e}+\left(\frac{d\,\gamma}{d\,k_\epsilon}\right)\frac{d\,\mathrm{R}}{d\,\gamma}$$

etc. etc. etc. etc.

Substituting the numerical values of $\frac{d\,\varpi}{d\,k_\epsilon}$ and the values (108) of $\frac{d\,\mathrm{R}_2}{d\,\varpi}$, etc., we have, for any one term of R_2,

$$\frac{d\,\mathrm{R}_2}{d\,k_\epsilon}=\mu_4 a^3 n\left\{\quad 2.023\,\mathrm{K}+1.011\frac{d\,\mathrm{K}}{d\,\varpi}-.0005\frac{d\,\mathrm{K}}{e\,d\,e}-.0005\frac{d\,\mathrm{K}}{\gamma\,d\,\gamma}\right\}\sin(\mathrm{N}\pm\mathrm{U})$$

$$+\mu_4 a^3 n\left\{\quad 2.023\,\mathrm{K}'+1.011\frac{d\,\mathrm{K}'}{d\,\varpi}-.0005\frac{d\,\mathrm{K}'}{e\,d\,e}-.0005\frac{d\,\mathrm{K}'}{\gamma\,d\,\gamma}\right\}\cos(\mathrm{N}\pm\mathrm{U})$$

$$\frac{d\,\mathrm{R}_2}{d\,k_\pi}=\mu_4 a^3 n\left\{-.0293\,\mathrm{K}-.0147\frac{d\,\mathrm{K}}{d\,\varpi}-0.526\frac{d\,\mathrm{K}}{e\,d\,e}-.0005\frac{d\,\mathrm{K}}{\gamma\,d\,\gamma}\right\}\sin(\mathrm{N}\pm\mathrm{U})$$

$$+\mu_4 a^3 n\left\{-.0293\,\mathrm{K}'-.0147\frac{d\,\mathrm{K}'}{d\,\varpi}-0.526\frac{d\,\mathrm{K}'}{e\,d\,e}-.0005\frac{d\,\mathrm{K}'}{\gamma\,d\,\gamma}\right\}\cos(\mathrm{N}\pm\mathrm{U})$$

$$\frac{d\,\mathrm{R}_2}{d\,k_\theta}=\mu_4 a^3 n\left\{+.0075\,\mathrm{K}+.0037\frac{d\,\mathrm{K}}{d\,\varpi}+.00004\frac{d\,\mathrm{K}}{e\,d\,e}-0.1250\frac{d\,\mathrm{K}}{\gamma\,d\gamma}\right\}\sin(\mathrm{N}\pm\mathrm{U})$$

$$+\mu_4 a^3 n\left\{+.0075\,\mathrm{K}'+.0037\frac{d\,\mathrm{K}'}{d\,\varpi}+.00004\frac{d\,\mathrm{K}'}{e\,d\,e}-0.1250\frac{d\,\mathrm{K}'}{\gamma\,d\gamma}\right\}\cos(\mathrm{N}\pm\mathrm{U})$$

Integrating, and substituting in the preceding expressions for $\delta\epsilon$, $\delta\pi$, and $\delta\theta$, we find that the important terms of these expressions may be expressed in the form

$$\begin{aligned}\delta\epsilon&=\epsilon_c\cos(\mathrm{N}\pm\mathrm{U})+\epsilon_s\sin(\mathrm{N}\pm\mathrm{U})\\ \delta\pi&=\pi_c\cos(\mathrm{N}\pm\mathrm{U})+\pi_s\sin(\mathrm{N}\pm\mathrm{U})\\ \delta\theta&=\theta_c\cos(\mathrm{N}\pm\mathrm{U})+\theta_s\sin(\mathrm{N}\pm\mathrm{U})\end{aligned}\qquad(112)$$

where we have

$$\epsilon_c=\mu_4 a^3\left\{(\quad 1.517\,i-.022\,i'+.005\,i'')\,\nu^2\mathrm{K}+(\quad 2.023\,\mathrm{K}+1.011\frac{d\,\mathrm{K}}{d\,\varpi})\,\nu\right\}$$

$$\epsilon_s=\mu_4 a^3\left\{(-1.517\,i+.022\,i'-.005\,i'')\,\nu^2\mathrm{K}'+(-2.023\,\mathrm{K}'-1.011\frac{d\,\mathrm{K}'}{d\,\varpi})\,\nu\right\}$$

$$\pi_c=\mu_4 a^3\left\{(-.0226\,i-.0100\,i'-.0124\,i'')\,\nu^2\mathrm{K}+(-.0293\,\mathrm{K}-.0147\frac{d\,\mathrm{K}}{d\,\varpi}\right.$$

$$\left.-.526\frac{d\,\mathrm{K}}{e\,d\,e})\,\nu\right\}\qquad(113)$$

$$\begin{aligned}
\pi_s &= \mu_4 a^3 \Big\{ \big(\ .0226\, i + .0100\, i' + .0124\, i'' \big) \nu^2 \mathrm{K}' + \big(+.0295\, \mathrm{K}' + .0147 \frac{d\mathrm{K}'}{d\alpha} \\
&\qquad + .526 \frac{d\mathrm{K}'}{c\,de} \big) \nu \Big\} \\
\theta_c &= \mu_4 a^3 \Big\{ \big(\ .0057\, i - .0124\, i' + .0019\, i'' \big) \nu^2 \mathrm{K} + \big(\ .0074\, \mathrm{K} + .0037 \frac{d\mathrm{K}}{d\alpha} \\
&\qquad - .1250 \frac{d\mathrm{K}}{\gamma\, d\gamma} \big) \nu \Big\} \\
\theta_s &= \mu_4 a^3 \Big\{ \big(-.0057\, i + .0124\, i' - .0019\, i'' \big) \nu^2 \mathrm{K}' + \big(-.0074\, \mathrm{K}' - .0037 \frac{d\mathrm{K}'}{d\alpha} \\
&\qquad + .1250 \frac{d\mathrm{K}'}{\gamma\, d\gamma} \big) \nu \Big\}
\end{aligned} \tag{113}$$

These expressions do not apply to the case of $\mathrm{N} \pm \mathrm{U} = 0$, since ν will then become infinite. In this case i, i', i'' all vanish, as already shown, and the values of $\frac{d\varepsilon}{dt}$, etc., reduce to constants. Integration then gives

$$\begin{aligned}
\delta\varepsilon &= \mu_4 a^3 \Big\{ -2.023\, \mathrm{K}'_0 - 1.011 \frac{d\mathrm{K}'_0}{d\alpha} + .0005 \frac{d\mathrm{K}'_0}{e\,de} \Big\} nt \\
\delta\pi &= \mu_4 a^3 \Big\{ \ .0293\, \mathrm{K}'_0 + .0147 \frac{d\mathrm{K}'_0}{d\alpha} + 0.526 \frac{d\mathrm{K}'_0}{e\,de} \Big\} nt \\
\delta\theta &= \mu_4 a^3 \Big\{ -.0074\, \mathrm{K}'_0 - .0037 \frac{d\mathrm{K}'_0}{d\alpha} + 0.125 \frac{d\mathrm{K}'_0}{\gamma\, d\gamma} \Big\} nt
\end{aligned} \tag{114}$$

We have next to find accurate or approximate values of the various quantities which enter in these expressions. Let us begin with the quantities $\varkappa_1$, $\varkappa_2$, etc., which enter into the values of K. The quantities are by (104) the coefficients of cos N or sin N in the development of $\frac{x^2}{a^2}$, $\frac{y^2}{a^2}$, etc., which powers and products are to be derived from the values of x, y, and z in (81) by the substitution of the analytical values of k_1 and k_2 given in the same section. If we take the plane of the ecliptic at the mean epoch for that of xy, the value of γ' will be of the order of the disturbing forces, and may be neglected entirely when attention is confined to quantities of the first order relatively to the disturbing forces. Putting $\gamma' = 0$, the values of x, y, and z become

$$\begin{aligned}
\frac{x}{a} &= k_1 \cos \Big\{ (1 + i + j)\, l - i\pi - (j + j')\, \theta + (i' + j')\, l' - i'\, \pi' \Big\} \\
&\quad + k_2 \cos \Big\{ (1 - i - j)\, l + i\pi + (j + j')\, \theta - (i' + j')\, l' + i'\, \pi' \Big\} \\
\frac{y}{a} &= k_1 \sin \Big\{ (1 + i + j)\, l - i\pi - (j + j')\, \theta + (i' + j')\, l' - i'\, \pi' \Big\} \\
&\quad + k_2 \sin \Big\{ (1 - i - j)\, l + i\pi + (j + j')\, \theta - (i' + j')\, l' + i'\, \pi' \Big\} \\
\frac{z}{a} &= c \sin \Big\{ (i_1 + j_1)\, l - i_1\, \pi - (j_1 + j'_1)\, \theta + (i'_1 + j'_1)\, l' - i'_1\, \pi' \Big\}
\end{aligned}$$

The position of the axis of x being entirely arbitrary we may suppose all the longitudes to be counted from the solar perigee. This will be effected by putting $\pi'=0$, which will give $l'=g'$. Thus, x, y, z, etc., will be expressed in terms of the four varying angles

$$l,\ \pi,\ \theta,\ g',$$

the three quantities l, π, and θ representing the mean longitude of the Moon, and the longitudes of its perigee and node, all counted from the solar perigee, and g' the mean anomaly of the Sun or of the Earth. We shall first give the development of x, y, and z, retaining the solar perigee, π', so that the axis of X remains arbitrary. The first column gives the indices i, i', j, and j', or the coefficients of g, g', λ, and λ'. The next gives the coefficients of l, π, θ, l', and π', derived from those indices. We then have the coefficients k_1 and k_2, which, being multiplied by the *cosines* of the angles, give the terms of $\frac{x}{a}$, and by the *sines* the terms of $\frac{y}{a}$

i,	i'	j,	j'	l,	π,	θ,	l'	π'	k_1 and k_2
0	0	0	0	1	0	0	0	0	$1-\frac{1}{2}e^2-\gamma^2-\frac{1}{6}m^2+\frac{1}{2}e^2\gamma^2-\frac{931}{384}e^2m^2$
1	0	0	0	2	-1	0	0	0	$+\frac{1}{2}e-\frac{3}{8}e^3-\frac{1}{2}e\gamma^2+\frac{7}{24}em^2$
				0	1	0	0	0	$-\frac{3}{2}e+\frac{3}{2}e\gamma^2+\frac{5}{8}em^2$
2	0	0	0	3	-2	0	0	0	$+\frac{3}{8}e^2-\frac{3}{8}e^4-\frac{3}{8}e^2\gamma^2+\frac{7}{32}e^2m^2$
				-1	2	0	0	0	$+\frac{1}{8}e^2+\frac{1}{24}e^4+\frac{9}{8}e^2\gamma^2-\frac{5}{96}e^2m^2$
3	0	0	0	4	-3	0	0	0	$+\frac{1}{3}e^3$
				-2	3	0	0	0	$+\frac{1}{24}e^3$
-1	1	0	0	0	1	0	1	-1	$-\frac{27}{16}ee'm$
				2	-1	0	-1	1	$+\frac{33}{16}ee'm$
0	1	0	0	1	0	0	1	-1	$-\frac{3}{2}e'm+\frac{3}{4}e'm^2$
				1	0	0	-1	1	$+\frac{3}{2}e'm+\frac{3}{4}e'm^2$

i	i'	j	j'	l	π	θ	l'	π'	k_1 and k_2
1	1	0	0	2	−1	0	1	−1	$-\frac{33}{16}ee'm$
				0	1	0	−1	1	$+\frac{27}{16}ee'm$
−1	0	2	0	2	1	−2	0	0	$-\frac{5}{4}e\gamma^2$
				0	−1	2	0	0	$+\frac{9}{4}e\gamma^2$
0	0	2	0	3	0	−2	0	0	$\left(-\frac{15}{8}e^2\gamma^2\right)$
				−1	0	2	0	0	$+\gamma^2$
1	0	2	0	4	−1	−2	0	0	0
				−2	1	2	0	0	$+\frac{1}{2}e\gamma^2$
−2	0	2	−2	1	2	0	−2	0	$\frac{15}{32}e^2m$
				1	−2	0	2	0	$-\frac{75}{32}e^2m$
−1	0	2	−2	2	1	0	−2	0	$\frac{15}{16}em+\frac{337}{64}em^2$
				0	−1	0	2	0	$-\frac{45}{16}cm-\frac{735}{64}em^2$
0	0	2	−2	3	0	0	−2	0	$+\frac{3}{16}m^2+\frac{7}{8}m^3+\frac{45}{32}e^2m$
				−1	0	0	2	0	$-\frac{19}{16}m^2-\frac{97}{24}m^3+\frac{15}{32}e^2m+\frac{3}{4}\gamma^2m$
1	0	2	−2	4	−1	0	−2	0	$+\frac{3}{8}em^2$
				−2	1	0	2	0	$-\frac{1}{16}em^2$
−1	−1	2	−2	2	1	0	−3	1	$+\frac{35}{16}ee'm$
				0	−1	0	3	−1	$-\frac{105}{16}ee'm$
0	−1	2	−2	3	0	0	−3	1	$+\frac{21}{32}e'm^2$
				−1	0	0	3	−1	$-\frac{133}{32}e'm^2$

i	i'	j	j'	l	π	θ	l'	π'	k_1 and k_2
-1	1	2	-2	2	1	0	-1	-1	$-\frac{15}{16}ee'm$
				0	-1	0	1	1	$+\frac{45}{16}ee'm$
0	1	2	-2	3	0	0	-1	-1	$-\frac{3}{32}e'm^2$
				-1	0	0	1	1	$+\frac{19}{32}e'm^2$
0	0	0	2	1	0	-2	2	0	$-\frac{3}{2}\gamma^2 m$
				1	0	2	-2	0	$+\frac{3}{4}\gamma^2 m$
0	0	1	-1	2	0	0	-1	0	$-\frac{15}{32}\alpha m$
				0	0	0	1	0	$+\frac{45}{32}\alpha m$
0	1	1	-0	2	0	0	0	-1	$+\frac{5}{8}e'\alpha$
				0	0	0	0	1	$-\frac{15}{8}e'\alpha$
0	2	0	0	1	0	0	2	-2	$-\frac{9}{8}e'^2 m$
				1	0	0	-2	2	$+\frac{9}{8}e'^2 m$

$$\frac{z}{a} = \Sigma\, c \sin \mathrm{N}$$

i	i'	j	j'	l	π	θ	l'	π'	c
-2	0	1	0	-1	2	-1	0	0	$-e^2\gamma$
-1	0	1	0	0	1	-1	0	0	$-3e\gamma$
0	0	1	0	1	0	-1	0	0	$+2\gamma-e^2\gamma-\gamma^3-\frac{1}{3}\gamma m^2$
1	0	1	0	2	-1	-1	0	0	$+e\gamma$
2	0	1	0	3	-2	-1	0	0	$+\frac{3}{4}e^2\gamma$
0	-1	1	0	1	0	-1	-1	1	$+\frac{3}{4}e'\gamma m$

$$\frac{z}{a}=\Sigma\, c \sin \mathrm{N}\text{—(Continued.)}$$

i	i'	j	j'	l	π	θ	l'	π'	c
0	1	1	0	1	0	−1	1	−1	$-\frac{3}{4}e'\gamma m$
−1	0	3	−2	2	1	−1	−2	0	$+\frac{15}{8}e\gamma m$
0	0	3	−2	3	0	−1	−2	0	$+\frac{3}{8}\gamma m^2$
0	−1	1	−2	1	0	1	−3	1	$+\frac{7}{4}e'\gamma m$
−1	0	1	−2	0	1	1	−2	0	$+\frac{9}{2}e\gamma m$
0	0	1	−2	1	0	1	−2	0	$+\frac{3}{4}\gamma m+\frac{41}{16}\gamma m^2$
1	0	1	−2	2	−1	1	−2	0	$+\frac{3}{8}e\gamma m$
0	1	1	−2	1	0	1	−1	−1	$-\frac{3}{4}e'\gamma m$

We next form the products x^2, y^2, z^2, $x\,y$, $y\,z$, $x\,z$, so as to obtain the expressions for κ_1, κ_2, κ_3, κ_4, κ_5, κ_6 defined in (104) and used in (107). The complete values to quantities of the third order, inclusive, are given in the following tables. We now replace l, π, θ, l' by the quantities

$$(l)= l-\pi'$$
$$(\pi)=\pi-\pi'$$
$$(\theta)=\theta-\pi'$$
$$g' = l'-\pi'$$

omitting the multiplier of π' entirely, which will be equivalent to supposing all the longitudes counted from the Sun's perihelion.

The computation stops at terms of the third order, because this is about the limit of possibly sensible terms of long period, and our immediate object is rather to detect the few such terms that may possibly exist than to make an accurate computation of their values. If necessary a more accurate determination of them can then be made after they are found.

$(l)(\pi)(\theta)\ g'$	Values of $\varkappa_1$	Values of $\varkappa_2$
0 0 0 0	$\frac{1}{2}+\frac{3}{4}e^2-\gamma^2-\frac{1}{6}m^2-\frac{3}{2}e^2\gamma^2+\gamma^4$ $+\frac{969}{768}e^2m^2+\frac{37}{192}\gamma^2m^2-\frac{1}{4}e'^2m^2$ $+\frac{407}{288}m^4+\frac{4995}{512}e^2m^3$	$\frac{1}{2}+\frac{3}{4}e^2-\gamma^2-\frac{1}{6}m^2-\frac{3}{2}e^2\gamma^2+\gamma^4$ $+\frac{969}{768}e^2m^2+\frac{37}{192}\gamma^2m^2-\frac{1}{4}e'^2m^2$ $+\frac{407}{288}m^2+\frac{4995}{512}e^2m^3$
0 0 0 1	$+\frac{67}{32}e'm^2$	$+\frac{29}{32}e'm^2$
0 0 0 2	$-\frac{19}{16}m^2-\frac{97}{24}m^3+\frac{75}{16}e^2m+\frac{3}{4}\gamma^2m$	$+\frac{19}{16}m^2+\frac{97}{24}m^3-\frac{75}{16}e^2m-\frac{3}{4}\gamma^2m$
0 0 0 3	$-\frac{133}{32}e'm^2$	$+\frac{133}{32}e'm^2$
0 0 2—2	$-\frac{3}{4}\gamma^2m$	$-\frac{3}{4}\gamma^2m$
0 0 2 0	$+\gamma^2$	$-\gamma^2$
0 2 0—2	$+\frac{45}{16}e^2m$	$+\frac{45}{16}e^2m$
0 2 0 0	$+\frac{5}{4}e^2-\frac{5}{4}e^2\gamma^2-\frac{97}{96}e^2m^2$	$-\frac{5}{4}e^2+\frac{5}{4}e^2\gamma^2+\frac{97}{96}e^2m^2$
1 0 0—1	$+\frac{15}{16}\alpha m$	$+\frac{15}{16}\alpha m$
1 0 0 0	$-\frac{5}{4}e'\alpha$	$+\frac{5}{8}e'\alpha$
1 0 0 1	$+\frac{45}{32}\alpha m$	$-\frac{45}{32}\alpha m$
1—3 0 0	$-\frac{7}{48}e^3$	$+\frac{7}{48}e^3$
1—1 0—2	$+\frac{55}{32}em^2$	$-\frac{55}{32}em^2$
1—1 0—1	$-\frac{21}{8}ee'm$	$-\frac{21}{8}ee'm$
1—1 0 0	$-e+\frac{1}{8}e^3+2e\gamma^2+\frac{13}{12}em^2$	$-e+\frac{1}{8}e^3+2e\gamma^2+\frac{13}{12}em^2$

(l)	(π)	(θ)	g'	Values of x_1	Values of x_2
1	−1	0	1	$+\frac{87}{16}ee'm$	$-\frac{3}{16}ee'm$
1	−1	0	2	$-\frac{45}{16}em-\frac{773}{64}em^2$	$+\frac{45}{16}em+\frac{773}{64}em^2$
1	−1	0	3	$-\frac{105}{16}ee'm$	$+\frac{105}{16}ee'm$
1	−1	−2	0	$-e\gamma^2$	$+e\gamma^2$
1	−1	2	0	$+\frac{11}{4}e\gamma^2$	$-\frac{11}{4}e\gamma^2$
1	1	0	−3	$-\frac{35}{8}ee'm$	$-\frac{35}{8}ee'm$
1	1	0	−2	$-\frac{15}{8}em-\frac{139}{32}em^2$	$-\frac{15}{8}em-\frac{139}{32}em^2$
1	1	0	−1	. . . .	$+\frac{39}{16}ee'm$
1	1	0	0	$-\frac{3}{2}e+\frac{13}{16}e^3+3e\gamma^2+\frac{7}{8}em^2$	$+\frac{3}{2}e-\frac{13}{16}e^3-3e\gamma^2-\frac{7}{8}em^2$
1	1	0	1	$+\frac{9}{16}ee'm$	$-\frac{9}{16}ee'm$
1	1	−2	0	$-\frac{1}{2}e\gamma^2$	$-\frac{1}{2}e\gamma^2$
2	0	0	−3	$-\frac{7}{2}e'm^2$	$-\frac{7}{2}e'm^2$
2	0	0	−2	$-m^2-\frac{19}{6}m^3+\frac{15}{32}e^2m+\frac{3}{4}\gamma^2m$	$-m^2-\frac{19}{6}m^3+\frac{15}{32}e^2m+\frac{3}{4}\gamma^2m$
				$+\frac{9}{8}e'^2m$	$-\frac{9}{8}e'^2m$
2	0	0	−1	$\frac{3}{2}e'm+\frac{5}{4}e'm^2$	$-\frac{3}{2}e'm-\frac{1}{4}e'm^2$
2	0	0	0	$\frac{1}{2}-2e^2-\gamma^2-\frac{1}{6}m^2$	$-\frac{1}{2}+\frac{5}{4}e^2+\gamma^2+\frac{1}{6}m^2$
2	0	0	1	$-\frac{3}{2}e'm+\frac{3}{4}e'm^2$	$+\frac{3}{2}e'm-\frac{3}{4}e'm^2$

(l)	(π)	(θ)	g'	Values of x_1	Values of x_2
2	0	0	2	$-\frac{45}{32}e^2m$	$\frac{9}{8}e'^2m-\frac{45}{32}e^2m$
2	0	—2	0	$+\gamma^2$	$+\gamma^2$
2	0	—2	2	$-\frac{3}{2}\gamma^2m$	$+\frac{3}{2}\gamma^2m$
2	0	2	—2	$+\frac{3}{4}\gamma^2m$	$-\frac{3}{4}\gamma^2m$
2	—2	0	0	$-\frac{1}{4}e^2$	$-\frac{1}{4}e^2$
2	—2	0	2	$-\frac{15}{4}e^2m$	$+\frac{15}{4}e^2m$
2	2	0	—2	$-\frac{15}{16}e^2m$	$+\frac{15}{16}e^2m$
3	0	0	—1	$-\frac{15}{32}\alpha m$	$+\frac{15}{32}\alpha m$
3	0	0	0	$-\frac{5}{4}e'\alpha$	$-\frac{5}{8}e'\alpha$
3	—1	0	—2	$-\frac{9}{16}e^2m$	$-\frac{9}{16}em^2$
3	—1	0	—1	$+\frac{33}{16}ee'm$	$-\frac{45}{16}ee'm$
3	—1	0	0	$-\frac{1}{2}e-\frac{19}{16}e^3-e\gamma^2+\frac{5}{24}em^2$	$-\frac{1}{2}e+\frac{19}{16}e^3+e\gamma^2-\frac{5}{24}em^2$
3	—1	0	1	$-\frac{21}{16}ee'm$	$+\frac{45}{16}ee'm$
3	—1	—2	—0	$+e\gamma^2$	$+e\gamma^2$
3	1	0	—3	$+\frac{35}{16}ee'm$	$-\frac{35}{16}ee'm$
3	1	0	—2	$+\frac{15}{16}em+\frac{319}{64}em^2$	$-\frac{15}{16}em-\frac{319}{64}em^2$
3	1	0	—1	$-\frac{3}{16}ee'm$	$+\frac{15}{16}ee'm$

$(l)(\pi)(\theta)\,g'$	Values of x_1	Values of x_2
3 1 −2 0	$-\frac{5}{4}e\gamma^2$	$+\frac{5}{4}e\gamma^2$
3 −3 0 0	$-\frac{1}{8}e^3$	$-\frac{1}{8}e^3$
4 0 0 −3	$+\frac{21}{32}e'm^2$	$-\frac{21}{32}e'm^2$
4 0 0 −2	$+\frac{3}{16}m^2+\frac{21}{24}m^3+\frac{15}{8}e^2m$	$-\frac{3}{16}m^2-\frac{21}{24}m^3-\frac{15}{8}e^2m$
4 0 0 −1	$-\frac{3}{32}ee'm$	$+\frac{3}{32}e'm^2$
4 −2 0 0	$+\frac{3}{8}e^2$	$-\frac{1}{2}e^2$
	Values of x_3	Values of x_4
0 0 0 0	$2\gamma^2$	0
0 0 0 1		$+\frac{19}{32}e'm^2$
0 0 0 2		$-\frac{19}{16}m^2-\frac{97}{24}m^3+\frac{75}{16}e^2m+\frac{3}{4}\gamma^2m$
0 0 0 3		$-\frac{133}{32}e'm^2$
0 0 2 −2	$+\frac{3}{2}\gamma^2m$	
0 0 2 0		$+\gamma^2$
0 2 0 −2		
0 2 0 0		$+\frac{5}{4}e^2-\frac{5}{4}e^2\gamma^2-\frac{97}{96}e^2m^2$
1 0 0 −1		
1 0 0 0		$-\frac{15}{8}e'\alpha$
1 0 0 1		$+\frac{45}{32}\alpha m$

$(l)(\pi)(\theta)\,g'$	Values of x_3	Values of x_4
1 −1 0 −2		$-\frac{55}{32}em^2$
1 −1 0 −1		
1 −1 0 0	$-4e\gamma^2$	
1 −1 0 1		$+\frac{45}{16}ee'm$
1 −1 0 2		$-\frac{45}{16}em-\frac{773}{64}em^2$
1 −1 0 3		$-\frac{105}{16}ee'm$
1 −1 −2 0		$-e\gamma^2$
1 −1 2 0		$+\frac{11}{4}e\gamma^2$
1 1 0 −3		
1 1 0 −2		
1 1 0 −1		$-\frac{9}{16}ee'm$
1 1 0 0		$-\frac{3}{2}e+\frac{13}{16}e^3+3e\gamma^2+\frac{7}{8}em^2$
1 1 0 1		$+\frac{9}{16}ee'm$
1 1 −2 0	$+6e\gamma^2$	
1 −3 0 0		$+\frac{7}{48}e^3$
2 0 0 −3		
2 0 0 −2	$-\underline{3}\,\gamma^2 m$	$+\frac{9}{8}e'^2m$
2 0 0 −1		$+\frac{3}{2}e'm+\frac{3}{4}e'm^2$

(l)	(π)	(θ)	g'	Values of $_{3}$	Values of x_4
2	0	0	0		$+\frac{1}{2}-\frac{5}{4}e^2-\gamma^2-\frac{1}{6}m^2$
2	0	0	1		$-\frac{3}{2}e'm+\frac{3}{4}e'm^2$
2	0	0	2		$-\frac{9}{8}e'^2m$
2	0	−2	0	$-2\gamma^2$	
2	0	−2	2		$-\frac{3}{2}\gamma^2m$
2	0	2	−2		$+\frac{3}{4}\gamma^2m$
2	−2	0	0		
2	−2	0	2		$-\frac{15}{4}e^2m$
2	2	0	−2		$-\frac{15}{16}e^2m$
3	0	0	−1		$-\frac{15}{32}\alpha m$
3	0	0	0		$+\frac{5}{8}e'\alpha$
3	−1	0	−2		
3	−1	0	−1		$+\frac{45}{16}ee'm$
3	−1	0	0		$+\frac{1}{2}e-\frac{19}{16}e^3-e\gamma^2+\frac{5}{24}em^2$
3	−1	0	1		$-\frac{45}{16}ee'm$
3	−1	−2	0	$-2e\gamma^2$	
3	1	0	−3		$+\frac{35}{16}ee'm$

$(l)\,(\pi)\,(\theta)\,g'$	Values of $\varkappa_3$	Values of $\varkappa_4$
3 1 0 −2		$+\frac{15}{16}em+\frac{319}{64}em^2$
3 1 0 −1		$-\frac{15}{16}ee'm$
3 1 −2 0		$-\frac{5}{4}e\gamma^2$
3 −3 0 0		
4 0 0 −3		
4 0 0 −2		
4 0 0 −1		
4 −2 0 0		$+\frac{3}{8}e^2$
	Values of $2\,\varkappa_5$	Values of $2\,\varkappa_6$
0 0 1 −3	$+\frac{7}{4}e'\gamma m$	$+\frac{7}{4}e'\gamma m$
0 0 1 −2	$+\frac{3}{4}\gamma m+\frac{79}{16}\gamma m^2$	$\frac{3}{4}\gamma m$
0 0 1 −1	$+\frac{3}{2}e'\gamma m$	$-3e'\gamma m$
0 0 1 0	$-2\gamma-3e^2\gamma+5\gamma^3+\frac{2}{3}\gamma m^2$	$-2\gamma-3e^2\gamma+5\gamma^3+\frac{2}{3}\gamma m^2$
0 0 1 1	$-\frac{9}{4}e'\gamma m$	$+\frac{9}{4}e'\gamma m$
0 2 −1 0	$-\frac{15}{4}e^2\gamma$	$+\frac{15}{4}e^2\gamma$
1 −1 −1 0	$-2e\gamma$	$-2e\gamma$
1 −1 −1 2	$-\frac{21}{2}e\gamma m$	$-\frac{21}{2}e\gamma m$
1 −1 1 −2	$-\frac{3}{4}e\gamma m$	$-\frac{3}{4}e\gamma m$

$(l)(\pi)(\theta)\ g'$	Values of $2\,\varkappa_5$	Values of $2\,\varkappa_6$
$1\ -1\ \ 1\ \ 0$	$+2e\gamma$	$+2e\gamma$
$1\ \ 1\ -1\ -2$	$-\frac{15}{4}e\gamma m$	$-\frac{15}{4}e\gamma m$
$1\ \ 1\ -1\ \ 0$	$+6e\gamma$	$-6e\gamma$
$1\ \ 1\ \ 1\ -2$	$-\frac{3}{2}e\gamma m$	$+\frac{3}{2}e\gamma m$
$2\ -2\ -1\ \ 0$	$-\frac{1}{2}e^2\gamma$	$-\frac{1}{2}e^2\gamma$
$2\ -2\ \ 1\ \ 0$	$+\frac{7}{4}e^2\gamma$	$+\frac{7}{4}e^2\gamma$
$2\ \ 0\ -3\ \ 0$	$+2\gamma^3$	$+2\gamma^3$
$2\ \ 0\ -1\ -2$	$-2\gamma m^2$	$-2\gamma m^2$
$2\ \ 0\ -1\ -1$	$-\frac{15}{4}e'\gamma m$	$+\frac{15}{4}e'\gamma m$
$2\ \ 0\ -1\ \ 0$	$-2\gamma+5e^2\gamma+3\gamma^3+\frac{2}{3}\gamma m^2$	$+2\gamma-5e^2\gamma-3\gamma^3-\frac{2}{3}\gamma m^2$
$2\ \ 0\ -1\ \ 1$	$-\frac{15}{4}e'\gamma m$	$-\frac{15}{4}e'\gamma m$
$2\ \ 0\ \ 1\ -3$	$-\frac{7}{4}e'\gamma m$	$+\frac{7}{4}e'\gamma m$
$2\ \ 0\ \ 1\ -2$	$-\frac{3}{4}\gamma m-\frac{35}{16}\gamma m^2$	$+\frac{3}{4}\gamma m+\frac{35}{16}\gamma m^2$
$2\ \ 0\ \ 1\ -1$	$+\frac{3}{4}e'\gamma m$	$-\frac{3}{4}e'\gamma m$
$3\ -1\ -1\ \ 0$	$-2e\gamma$	$+2e\gamma$
$3\ -1\ \ 1\ -2$	$-\frac{3}{4}e\gamma m$	$+\frac{3}{4}e\gamma m$
$3\ \ 1\ -1\ -2$	$-\frac{15}{4}e\gamma m$	$+\frac{15}{4}e\gamma m$
$4\ -2\ -1\ \ 0$	$-2e^2\gamma$	$+2e^2\gamma$
$4\ \ 0\ -1\ -2$	$-\frac{3}{4}\gamma m^2$	$+\frac{3}{4}\gamma m^2$

The values (107) of K and K′ and of such of their derivatives as enter into the equations (112) to (114) are next to be investigated. Owing to the great number of the separate factors $\varkappa_i$, K_i, and K'_i, and the small number of combinations which give rise to sensible inequalities, we shall adopt the course of deferring the reduction of the $\varkappa_i$ to numbers until we find what values of those quantities are required. It is, however, practically inconvenient to compute individual values of K and K′; we shall therefore show how all sensible values of these quantities may be found by developing the expressions (103) in numerical series.

Although the numerical method adopted in the next chapter seems to me that best adapted to the detection of inequalities of long period, it is quite possible that, after the argument of such an inequality is established, its coefficient can be best found by the method of analytic development.

CHAPTER III.

NUMERICAL DEVELOPMENTS FOR DETERMINING THE ACTION OF THE SEPARATE PLANETS ON THE MOON.

§ 17.

DEVELOPMENT OF THOSE FACTORS IN THE PERTURBATIVE FUNCTION WHICH DEPEND UPON THE COORDINATES OF THE SUN AND OF THE PLANET.

Having found the factors $\varkappa_i$ in the expression (107) we have next to find K_1, K_2, etc., as defined in (103), and used in (107). The first and most difficult operation under this head is the development of $\frac{1}{\rho^3}$ and $\frac{1}{\rho^5}$, ρ being the distance of the planet from the center of gravity of the Earth and Moon. It is necessary to carry this development to much higher multiples of the mean anomalies, and to higher powers of the eccentricities, than are necessary in computing the perturbations of the planets. The labor of computing the numerical values of the terms of the development from their analytical expressions would be enormous. I have therefore adopted the numerical method, in which the development is first effected in terms of the eccentric anomalies.

The original inventor of this method was, I believe, CAUCHY, who explained it in a series of short papers published in the *Comptes Rendus*, between 1843 and 1845. A systematic exposition of it, according to the plan of CAUCHY, is given by PUISEUX in *Annales de l'Observatoire Impériale de Paris, Tome VII*, where, however, it is applied only to the computation of particular terms which may give rise to inequalities of long period and not to the general development. The same volume contains a paper by BOURGET, in which it is applied to the general development of the perturbative function. The adaptation of the method to the numerical computation of the general development of the perturbative function and its derivatives is mainly due to HANSEN, who has developed it very copiously in his *Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten : Erste Abhandlung*, found in the *Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften, Band III, Leipzig, 1857*. HANSEN's exposition of the method is so well adapted to numerical computation that I have generally followed it, and the papers of both HANSEN and PUISEUX are so copious that it is not necessary to present a detailed development of the formulæ employed. But, as I have found it convenient to adopt forms of computation not expressly given by HANSEN, which may possibly be useful in the general computation of planetary perturbations, I shall present a brief derivation of the formulæ as I have used them. As it is convenient to use a notation in some respects peculiar to the present chapter, we present that actually adopted. Let us put

f, f', the true anomalies of any two planets.
u u', their eccentric anomalies.
g g', their mean anomalies.
w, w', the distances of their perihelia from their common node.
ρ, their linear distance.
γ, the mutual inclination of their orbits.
φ, φ', their angles of eccentricity, so that $e = \sin\varphi$

We then have for the square of their distance

$$\rho^2 = r^2 + r'^2 - 2\,r\,r' \left\{ \cos(f+w)\cos(f'+w') + \cos\gamma \sin(f+w)\sin(f'+w') \right\}$$

If we substitute for r', $r'\cos f'$, and $r'\sin f'$ their values in terms of the eccentric anomaly

$$r' = a'(1 - e'\cos u')$$
$$r'\cos f' = a'(\cos u' - e')$$
$$r'\sin f' = a'\cos\varphi'\sin u'$$

and retain r, f, and w, we find that the value of ρ^2 may be put in the form

$$\rho^2 = \mathrm{H} - \mathrm{K}\cos\psi\cos u' - \mathrm{K}\sin\psi\sin u' + s\cos^2 u' \qquad (116)$$

where H, K $\cos\psi$, K $\sin\psi$, and s have the following values:

$$\mathrm{H} = r^2 + a'^2 + 2a'e'r \left\{ \cos w'\cos(f+w) + \cos\gamma\sin w'\sin(f+w) \right\}$$

$$\mathrm{K}\cos\psi = 2\,a'\,r \left\{ \cos w'\cos(f+w) + \cos\gamma\sin w'\sin(f+w) \right\} + 2\,e'\,a'^2$$

$$\mathrm{K}\sin\psi = 2\,a'\,r\cos\varphi' \left\{ -\sin w'\cos(f+w) + \cos\gamma\cos w'\sin(f+w) \right\}$$

$$s = a'^2\,e'^2$$

The computation of these quantities may be simplified by introducing the auxiliary quantities k_1, k_2, and the angles K_1 and K_2 determined by the equations

$$
\left.\begin{aligned}
k_1 \sin K_1 &= \cos \gamma \sin w' \\
k_1 \cos K_1 &= \cos w' \\
k_2 \sin K_2 &= \sin w' \\
k_2 \cos K_2 &= \cos \gamma \cos w'
\end{aligned}\right\} \qquad (117)
$$

We then have

$$
\left.\begin{aligned}
H &= r^2 + a'^2 + 2\, a'\, e'\, k_1\, r \cos (f + w - K_1) \\
K \cos \psi &= 2\, a'\, k_1\, r \cos (f + w - K_1) + 2\, e'\, a'^2 \\
K \sin \psi &= 2\, a'\, r\, k_2 \cos \varphi' \sin (f + w - K_2) \\
s &= e'^2\, a'^2
\end{aligned}\right\} \qquad (118)
$$

These four quantities are functions of the elements of both planets, and of the coordinates of the unaccented planet, but do not contain the coordinates of the accented planet. Thus, by the equation (116) we have the theorem:

If the position of a planet be given, the square of its distance from any other planet may be expressed as a rational and entire function of the sine and cosine of the eccentric anomaly of the latter.

On this principle is based the CAUCHY-HANSEN method of developing the perturbative function.

Both CAUCHY and HANSEN express the coefficients H, K cos ψ, etc., as functions of the eccentric anomaly u, but for our present purposes I have deemed it more convenient to retain the true anomaly f. Having computed them for any assumed value of f, we have next to transform the expression (116) into the form

$$
\rho^2 = \{ C - q \cos (Q - u') \} \, \{ 1 - q_1 \cos (Q + u') \} \qquad (119)
$$

Developing this product and comparing with (116) we find that C, Q, q, and q_1 are to be determined by the equations

$$
\left.\begin{aligned}
q_1 &= \frac{s}{q} \\
C &= H + s \sin^2 Q \\
q \cos Q &= K \cos \psi - C\, q_1 \cos Q \\
q \sin Q &= K \sin \psi + C\, q_1 \sin Q
\end{aligned}\right\} \qquad (120)
$$

Let us substitute in the last members of these equations the first approximate values of C, q, and Q, obtained by putting $q_1 = 0$, namely, H, K, and φ. We shall then have, for a second approximation, which includes the first powers of s only

$$
\left.
\begin{aligned}
q\cos Q &= \mathrm{K}\cos\psi\left(1-\frac{\mathrm{H}s}{\mathrm{K}^2}\right)\\
q\sin Q &= \mathrm{K}\sin\psi\left(1+\frac{\mathrm{H}s}{\mathrm{K}^2}\right)\\
q &= \mathrm{K}\left(1-\frac{\mathrm{H}s}{\mathrm{K}^2}\cos 2\psi\right)\\
\sin Q &= \sin\psi\left(1+\frac{2\,\mathrm{H}s}{\mathrm{K}^2}\cos^2\psi\right)
\end{aligned}
\right\}\qquad (121)
$$

The substitution of the last two expressions in the first two of (120) will give the following values of q and C, true to terms of the second order in s

$$
\left.
\begin{aligned}
q_1 &= \frac{s}{\mathrm{K}}\left(1+\frac{\mathrm{H}s}{\mathrm{K}^2}\cos 2\psi\right)\\
\mathrm{C} &= \mathrm{H}+s\left(1+4\frac{\mathrm{H}s}{\mathrm{K}^2}\cos^2\psi\right)\sin^2\psi
\end{aligned}
\right\}\qquad (122)
$$

It is most convenient to compute C from this formulæ before proceeding to compute q and Q. If we retain C in the last members of the last two equations (120) and substitute for q_1 its value just given, we shall have q cos Q and q sin Q to terms of the second order in s. First, we obtain from (121)

$$\mathrm{C}q_1\cos Q = \frac{\mathrm{C}q_1}{q}\mathrm{K}\cos\psi\left(1-\frac{\mathrm{H}s}{\mathrm{K}^2}\right)$$

$$\mathrm{C}q_1\sin Q = \frac{\mathrm{C}q_1}{q}\mathrm{K}\sin\psi\left(1+\frac{\mathrm{H}s}{\mathrm{K}^2}\right)$$

But from (121) and (122) we have to terms of the second order in s

$$\frac{q_1}{q} = \frac{s}{\mathrm{K}^2}\left(1+\frac{2\,\mathrm{H}s}{\mathrm{K}^2}\cos 2\psi\right)$$

which, being substituted in the last equations, gives

$$\mathrm{C}\,q_1\cos Q = \frac{\mathrm{C}s}{\mathrm{K}}\cos\psi\left(1+\frac{\mathrm{H}s}{\mathrm{K}^2}(2\cos 2\psi)-1\right)$$

$$= \frac{\mathrm{C}s}{\mathrm{K}}\cos\psi\left(1+\frac{\mathrm{H}s}{\mathrm{K}^2}(1-4\sin^2\psi)\right)$$

$$\mathrm{C}\,q_1\sin Q = \frac{\mathrm{C}s}{\mathrm{K}}\sin\psi\left(1+\frac{\mathrm{H}s}{\mathrm{K}^2}(1+2\cos 2\psi)\right)$$

$$= \frac{\mathrm{C}s}{\mathrm{K}}\sin\psi\left(1-\frac{\mathrm{H}s}{\mathrm{K}^2}(1-4\cos^2\psi)\right)$$

The substitution of these expressions in the last two equations (120) gives for the definitive values of q cos Q and q sin Q to terms of the second order in s

$$\begin{aligned} q\cos Q &= K\cos\psi\left\{1 - s\frac{C}{K^2}\left(1 + \frac{Hs}{K^2} - \frac{4Hs}{K^2}\sin^2\psi\right)\right\} \\ q\sin Q &= K\sin\psi\left\{1 + s\frac{C}{K^2}\left(1 - \frac{Hs}{K^2} + \frac{4Hs}{K^2}\cos^2\psi\right)\right\} \end{aligned} \qquad (123)$$

In these expressions the only quantities neglected are of the third order with respect to s, and therefore of the sixth order with respect to the eccentricity, and no case is likely to occur in which they can become sensible. The logarithmic computation of C, q, Q, and q_1 may be conveniently effected as follows:

Compute the five quantities s_1, F_1, F_2, F_3, and ζ by means of the equations

$$\begin{aligned} s_1 &= \frac{H}{K^2}s \\ F_1 &= 1 + 4s_1\cos^2\psi \\ F_2 &= 1 - 4s_1\sin^2\psi \\ F_3 &= 1 + s_1 \\ \zeta &= sF_1\sin^2\psi \end{aligned} \qquad (124)$$

We then have

$$\begin{aligned} C &= H + \zeta \\ q\sin Q &= \left(1 + s\frac{C}{K^2}\frac{F_1}{F_3}\right)K\sin\psi \\ q\cos Q &= \left(1 - s\frac{C}{K^2}F_2F_3\right)K\cos\psi \\ q_1 &= \frac{s}{q} \end{aligned} \qquad (125)$$

I put the factors into the form $1 \pm h$, because they are then very easily computed by **Zech**'s Table of addition and subtraction logarithms. By entering the addition table with the argument $\frac{1}{h}$ we take out the logarithm of $1 + h$, and by entering the subtraction table we take out $\log \frac{1}{1-h}$

Having thus, for any assumed position of the one planet, expressed the square of its distance from the other in the form (119), we have next to develop the negative and uneven powers of ρ. Here, again, I make use of the method of **Hansen**, which seems to be based upon the investigations of **Gauss**.* Let us put

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* The developments and rules on this and the next two pages are, to a great extent, superseded by those of Vol. III, pages 42–53 and 60–75. I have, however, retained them in the present form, as pertaining to the work, and as equally convenient with the more general ones of Vol. III, which apply to a wider range of cases. (See *post* p. 237.)

$$\rho_0^2 = C - q \cos(Q - u') \qquad (126)$$

so that we have

$$\rho^2 = \rho_0^2 \{ 1 - q_1 \cos(Q + u') \}$$

the last factor differing very little from unity.

We have then

$$\rho^{-n} = \rho_0^{-n} \{ 1 - q_1 \cos(Q + u') \}^{-\frac{n}{2}}$$

The first factor is to be developed into the form

$$\rho_0^{-n} = \frac{1}{2} \sum_{i=-\infty}^{i=\infty} b_n^{(i)} \cos i (Q - u')$$

If the cofficients $b_n^{(i)}$ be developed in powers of $\frac{q}{C}$ their general expressions will be

$$\frac{1}{2} b_n^{(i)} = \frac{n(n+2)(n+4) \ldots \ldots (n+2i-2)}{2.4.6 \ldots \ldots 2i} \frac{q^i}{C^i} F\left(\frac{n}{4} + \frac{i+1}{2}, \frac{n}{4} + \frac{i}{2}, i+1, \frac{q^2}{C^2}\right)$$

Here we use Gauss' notation for the hypergeometric series

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha \beta}{\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)} x^2 + \text{etc.}$$

We have also, by putting $i+1$ for i in the above value of b_n^i, and noticing that

$$F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$$

$$\frac{1}{2} b_n^{(i+1)} = \frac{n(n+2)(n+4) \ldots \ldots (n+2i)}{2.4.6 \ldots \ldots (2i+2)} \frac{q^{i+1}}{C^{i+1}} F\left(\frac{n}{4} + \frac{i+1}{2}, \frac{n}{4} + \frac{i}{2} + 1, i+2, \frac{q^2}{C^2}\right)$$

The ratio $\frac{b_n^{i+1}}{b_n^{(i)}}$ of two consecutive coefficients may now be developed in the form of a continued fraction by the method of Gauss, as given in his *Disquisitiones generales circa seriem infinitam, etc., Sectio Secunda.* Calling this ratio p_{i+1}, we have

$$p_i = \cfrac{\frac{n+2i-2}{4i} \frac{q}{C}}{1 - \cfrac{a}{1 - \cfrac{b}{1 - \cfrac{c}{\text{etc.}}}}}$$

where

$$a=\frac{(n+2i)(2i+2-n)}{16\,i\,(i+1)}\,\frac{q^2}{C^2}$$

$$b=\frac{(n+2i+2)(2i+4-n)}{16\,(i+1)(i+2)}\,\frac{q^2}{C^2}$$

$$c=\frac{(n+2i+4)(2i+6-n)}{16\,(i+2)(i+3)}\,\frac{q^2}{C^2}$$

etc.,

each term being formed from the preceding by increasing i by unity. We thus have the form given by HANSEN

$$p_i=\cfrac{F_i}{1-\cfrac{\lambda_{i+1}}{1-\cfrac{\lambda_{i+2}}{1-\cfrac{\lambda_{i+3}}{1-\text{etc.}}}}} \tag{127}$$

where

$$\left.\begin{aligned}\lambda_i&=\frac{(n+2i-2)(2i-n)}{16\,i\,(i-1)}\,\frac{q^2}{C^2}\\ F_i&=\frac{2i+n-2}{2i}\,\frac{q}{C}\end{aligned}\right\} \tag{128}$$

Owing to the slow convergence of the continued fraction (127) it is necessary to compute its value for the highest value of i independently. To derive the necessary equations we must develop $b_n^{(i)}$ in powers of α, a quantity to be determined by the equation

$$\frac{2\alpha}{1+\alpha^2}=\frac{q}{C}$$

the expression (126) being put in the form

$$\rho_0^2=k\,(1-2\,\alpha\cos(Q-u')+\alpha^2)$$

which is LA PLACE'S form for development.

I deem it unnecessary to go over the details of the derivation, since I have obtained the result given by HANSEN, whose formulæ may be most conveniently employed as follows:

Compute χ from the equation

$$\sin\chi=\frac{q}{c}$$

when we have

$$\alpha = \tan \frac{1}{2} \chi \qquad (129)$$

For the highest value of i to which it is deemed necessary to carry the development, compute γ_i by means of the continued fraction

$$\gamma_i = \cfrac{\sec^2 \frac{1}{2} \chi}{1 - \cfrac{a_1}{1 - \cfrac{b_1}{1 - \cfrac{a_2}{1 - \cfrac{b_2}{1 - \text{etc.}}}}}}$$

where

$$a_1 = \frac{n(2-n)}{4i(i+1)} \alpha^2 \qquad b_1 = \frac{(n+2i)(2i+2-n)}{4(i+1)(i+2)} \alpha^2$$

$$a_2 = \frac{(n+1)(4-n)}{4(i+2)(i+3)} \alpha^2 \qquad b_2 = \frac{(n+2i+2)(2i+4-n)}{4(i+3)(i+4)} \alpha^2$$

$$a_3 = \frac{(n+2)(6-n)}{4(i+4)(i+5)} \alpha^2 \qquad b_3 = \frac{(n+2i+4)(2i+6-n)}{4(i+5)(i+6)} \alpha^2$$

etc. etc.

Put also

$$\left. \begin{aligned} \theta_n^{(i)} &= \frac{n+2i-2}{4i} \\ \mu_n^{(i)} &= \frac{(2i-n)(n+2i-2)}{16i(i-1)} = \frac{2i-n}{4(i-1)} \theta_n^{(i)} \end{aligned} \right\} \qquad (130)$$

A general table of the values of $\theta_n^{(i)}$ and μ_n^i for all values of i and n commonly used is easy to form, and is given for $n=3$ and $n=5$ in the following section.

Then compute, for all values of i

$$\left. \begin{aligned} F_i &= \theta_n^{(i)} \sin \chi \\ \lambda_i &= \mu_n^{(i)} \sin^2 \chi \end{aligned} \right\} \qquad (131)$$

Commencing with the highest value of γ_i compute in succession all the other values by the formulæ

$$\gamma_i = \frac{1}{1 - \lambda_{i+1} \gamma_{i+1}} \qquad (132)$$

when we have

$$\left. \begin{aligned} p_i &= F_i \gamma_i \\ b_n^{(i)} &= p_i b_n^{(i-1)}. \end{aligned} \right\} \qquad (133)$$

The logarithms of γ_i in (132) can be computed with great facility by the table ot subtraction-logarithms. Then having all the values of log p_i we require the value of log $b_n^{(0)}$. This can be found most conveniently from RUNKLE's *Tables for determining the values of the Coefficients in the Perturbative Function of planetary motion which depend on the ratio of the mean distances.* (Smithsonian Contributions to Knowledge, Washington, 1855.) These tables give, for the argument α, the coefficients $b_s^{(i)}$ of cos $i\,\varphi$ in the development of

$$(1-2\alpha\cos\varphi+\alpha^2)^{-s}=\tfrac{1}{2}\Sigma b_s^{(i)}\cos i\,\varphi.$$

The development we actually seek is that of

$$(\mathrm{C}-q\cos\varphi)^{-\frac{n}{2}}=\tfrac{1}{2}\Sigma b_n^{(i)}\cos i\,\varphi.$$

To assimilate it to RUNKLE's development we must determine α and k so as to satisfy the equation

$$\mathrm{C}-q\cos\varphi=k\,(1+\alpha^2-2\,\alpha\cos\varphi)$$

Introducing the auxiliary χ,

$$\sin\chi=\frac{q}{\mathrm{C}},$$

we have

$$\begin{aligned}\alpha&=\tan\tfrac{1}{2}\chi\\ k&=\mathrm{C}\cos^2\tfrac{1}{2}\chi=\frac{q}{2\tan\frac{1}{2}\chi}\end{aligned}\qquad(134)$$

Then, comparing like terms in the two developments,

$$b_n^{(0)}=\sec^n\tfrac{1}{2}\chi\,\mathrm{C}^{-\frac{n}{2}}\,b_s^{(0)}$$

provided that $b_s^{(0)}$ be taken from RUNKLE's tables with the argument α. In the cases of $s=\frac{3}{2}$ and $s=\frac{5}{2}$ the quantities actually tabulated are,

$$\text{for } s=\tfrac{3}{2};\ (1-\alpha^2)^2\,b_s^{(0)}=\frac{\cos^2\chi}{\cos^4\frac{1}{2}\chi}\,b_s^{(0)}$$

$$\text{for } s=\tfrac{5}{2};\ (1-\alpha^2)^4\,b_s^{(0)}=\frac{\cos^4\chi}{\cos^8\frac{1}{2}\chi}\,b_s^{(0)}.$$

If, then, we represent by $\beta_{\frac{3}{2}}^{(0)}$ and $\beta_{\frac{5}{2}}^{(0)}$ the quantities actually taken from RUNKLE's tables when $s=\frac{3}{2}$ and $s=\frac{5}{2}$, we have

$$\begin{aligned}b_3^{(0)}&=\mathrm{C}^{-\frac{3}{2}}\sec^2\chi\cos\tfrac{1}{2}\chi\,\beta_{\frac{3}{2}}^{(0)}\\ b_5^{(0)}&=\mathrm{C}^{-\frac{5}{2}}\sec^4\chi\cos^3\tfrac{1}{2}\chi\,\beta_{\frac{5}{2}}^{(0)}\end{aligned}\qquad(135)$$

These quantities may also be computed with equal ease, though, perhaps, not always with equal accuracy, by the use of the arithmetico-geometrical mean. Representing this mean in the usual way by M we have, first,

$$b_1^{(0)} = \frac{2}{\sqrt{C}\,M(\cos\tfrac{1}{2}\chi,\ \sqrt{\cos\chi})} = \frac{2}{\sqrt{C\cos\chi}\,M\left(1,\ \dfrac{\cos\frac{1}{2}\chi}{\sqrt{\cos\chi}}\right)} \tag{136}$$

Then $b_3^{(0)}$ and $b_5^{(0)}$ may be obtained by the formulæ

$$\left.\begin{aligned} b_3^{(0)} &= \frac{b_1^{(0)}}{C - q\,p_3^{(1)}} = \frac{b_1^{(0)}}{C\,(1 - p_3^{(1)}\sin\chi)} \\ b_5^{(0)} &= \frac{b_3^{(0)}}{C - q\,p_5^{(1)}} = \frac{b_3^{(0)}}{C\,(1 - p_5^{(1)}\sin\chi)} \end{aligned}\right\} \tag{137}$$

When $p_3^{(1)} \sin\chi$ and $p_5^{(1)} \sin\chi$ approach to unity, these formulæ will give $b_3^{(0)}$ and $b_5^{(0)}$ with only very limited accuracy and others must be used.

The result of the preceding operations is that, for any assumed position of the unaccented planet in its orbit, we may develop the value of ρ_0^{-n} in the form

$$\rho_0^{-n} = \tfrac{1}{2} \sum_{-\infty}^{\infty} b_n^{(i)} \cos i\,(Q - u') \tag{138}$$

We now compute this development for a number of positions corresponding to equidistant values of the mean anomaly, g, of the planet, and thus obtain a separate value of $b_n^{(i)}$ and of Q for each value of the mean anomaly. Then, any functions of these quantities may be developed in a periodic series, proceeding according to the multiples of the mean anomaly by the well-known process of mechanical development. The most convenient functions to develop thus are $b_n^{(i)} \cos i\,(Q - g)$ and $b_n^{(i)} \sin i\,(Q - g)$. If we put

$$\left.\begin{aligned} Q' &= Q - g \\ A &= u' - g \end{aligned}\right\} \tag{139}$$

we have

$$\begin{aligned} \rho_0^{-n} &= \tfrac{1}{2} \Sigma\, b_n^{(i)} \cos(i\,Q' - i\,A) \\ &= \tfrac{1}{2} \Sigma\, b_n^{(i)} \cos i\,Q' \cos i\,A + \tfrac{1}{2} \Sigma\, b_n^{(i)} \sin i\,Q' \sin i\,A. \end{aligned}$$

Let us, then, develop $b_n^{(i)} \cos i\,Q'$ and $b_n^{(i)} \sin i\,Q'$ in a periodic series of the form

$$\left.\begin{aligned} b_n^{(i)} \cos i\,Q' &= c_0^{(i)} + c_1^{(i)} \cos g + c_2^{(i)} \cos 2g +, \text{etc.} \\ &\quad + s_1^{(i)} \sin g + s_2^{(i)} \sin 2g +, \text{etc.} \\ b_n^{(i)} \sin i\,Q' &= c'^{(i)}_0 + c'^{(i)}_1 \cos g + c'^{(i)}_2 \cos 2g +, \text{etc.} \\ &\quad + s'^{(i)}_1 \sin g + s'^{(i)}_2 \sin 2g +, \text{etc.} \end{aligned}\right\} \tag{140}$$

Then, let us compute values of $(a_{i,j})$ and $(b_{i,j})$ for all positive values of i and j from the formulæ

$$
\begin{array}{lll}
(a_{0,0})=\frac{1}{2}\,c_0^{(0)} & & \\[1em]
(a_{0,j}) \quad \frac{1}{2}\,c_j^{(0)} & (b_{0,j})=\frac{1}{2}\,s_i^{(0)} & \\[1em]
(a_{i,0})=\frac{1}{2}\,c_j^{(i)} & (b_{i,0})=c'^{(i)}_0 & \qquad (141)\\[1em]
(a_{i,j})=\frac{1}{2}(c_j^{(i)}-s'^{(i)}_j) & (b_{i,j})=\frac{1}{2}(c'^{(i)}_j+s_j^{(i)}) & \\[1em]
(a_{i,-j})=\frac{1}{2}(c_j^{(i)}+s'^{(i)}_j) & (b_{i,-j})=\frac{1}{2}(c'^{(i)}_j-s_j^{(i)}) &
\end{array}
$$

and we shall have the numerical expression for ρ_0^{-n} for all positions of the two planets by the formula

$$\rho_0^{-n}=\sum_{0}^{\infty}{}_i\sum_{-\infty}^{+\infty}{}_j\left\{(a_{i,j})\cos(i\mathrm{A}+jg)+(b_{i,j})\sin(i\mathrm{A}+jg)\right\} \qquad (142)$$

To obtain ρ^{-n}, this development is still to be multiplied by the factor

$$\mathrm{F}=\{1-q_1\cos(\mathrm{Q}+u')\}^{-\frac{2}{n}}=\{1-q_1\cos\mathrm{Q}'\cos(\mathrm{A}+2g)+q_1\sin\mathrm{Q}'\sin(\mathrm{A}+2g)\}^{-\frac{2}{n}}$$

Owing to the minuteness of the factor q_1 we may nearly always neglect its powers, and so put

$$\mathrm{F}=1+\frac{n}{2}q_1\cos\mathrm{Q}'\cos(\mathrm{A}+2g)-\frac{n}{2}q_1\sin\mathrm{Q}'\sin(\mathrm{A}+2g)$$

Let us develop as before $\frac{n}{2}q_1\cos\mathrm{Q}'$ and $\frac{n}{2}q_1\sin\mathrm{Q}'$ into a periodic series of the form

$$
\begin{aligned}
\frac{n}{2}q_1\cos Q' &= p_0+p_1\cos g+p_2\cos 2g+\text{etc.}\\
&\quad +q_1\sin g+q_2\sin 2g+\text{etc.}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (143)\\
\frac{n}{2}q_1\sin\mathrm{Q}' &= p'_0+p'_1\cos g+p'_2\cos 2g+\text{etc.}\\
&\quad +q'_1\sin g+q'_2\sin 2g+\text{etc.}
\end{aligned}
$$

and we shall have

$$
\begin{aligned}
\mathrm{F}=1&+p_0\cos(\mathrm{A}+2g)-p'_0\sin(\mathrm{A}+2g)\\
&+\Sigma\frac{1}{2}(p_i+q'_i)\cos\left(\mathrm{A}+(2+i)g\right)+\Sigma\frac{1}{2}(p_i-q'_i)\cos\left(\mathrm{A}+(2-i)g\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (144)\\
&-\Sigma\frac{1}{2}(p'_i-q_i)\sin\left(\mathrm{A}+(2+i)g\right)-\Sigma\frac{1}{2}(p'_i+q_i)\sin\left(\mathrm{A}+(2-i)g\right)
\end{aligned}
$$

Multiplying the series (142) and (144) term by term, we shall have the value of ρ^{-n} expressed in the form (142). The development will be given in terms of the mean anomaly of the one planet and the eccentric anomaly of the other.

The next process is to change it into a development in terms of the mean anomalies of both planets. This is done by means of the BESSELIAN transcendents, tables and formulæ for which are given by BESSEL in the *Berlin Memoirs* for 1824, while the mode of applying them is fully explained by HANSEN in his "*Auseinandersetzung.*"

Having thus effected the development of $\frac{1}{\rho^5}$ in terms of the mean anomalies of the Earth and planet, we have only to express x', y', z', the rectangular coordinates of the Sun relatively to the center of gravity of the Earth and Moon, and x_4, y_4, z_4, the rectangular coordinates of the planet relatively to the Sun, in terms of the respective mean anomalies. The factors depending on the coordinates of the planet and Sun will then be formed by multiplication. As the complete formation of these terms would be very laborious, we shall only actually develop those of which we find ourselves in need, and these can not be selected until we proceed with the more particular consideration of the action of the separate planets.

§ 18.

NUMERICAL DETAILS OF QUANTITIES DEPENDING ON THE POSITION OF THE PLANETS MERCURY, VENUS, AND MARS.

The computations of which the results are now given were made in the years 1871–'73, before I had prepared the more general discussion of the method found in Vol. III, Part I, Chapters II and IV. The latter does not, however, involve any material change in the method so far as the present application is concerned, where no deviation from HANSEN'S procedure is necessary. The object of the present section is to present such details as will enable the results to be verified, corrected, or reproduced with the least amount of labor.

The computations were made with more or less fullness for Mercury, Venus, and Mars. In the case of Mercury, however, the development was in error from the use of the longitude of the solar perigee, when that of the Earth's perihelion should have been used. As this error would but slightly change the larger terms in the development, and therefore would not invalidate the general conclusion that the action of Mercury produce no inequalities of importance in the Moon's motion, it was not corrected by a recomputation.

In the case of Mars the developments were not completed beyond the point where the eccentric has to be replaced by the mean anomaly, because the order of magnitude of the coefficients in the development according to the eccentric anomaly would suffice to draw general conclusions as to the possibility of hidden terms of long period.

In making the computations the first question is of which eccentric anomaly the development shall be a function. Theoretically this is indifferent; in practice, however, it will be convenient to choose the eccentric anomaly of the orbit having the smaller eccentricity, because the BESSELIAN transformation to mean anomaly will then be simpler, as will the change of ρ_0^{-n} into ρ^{-n}.

It will then be better to assume special equidistant values of the mean anomaly of the other planet rather than of its eccentric anomaly, because the necessity of a transformation to mean anomaly of this planet will then be avoided. For each position of the planet thus arising the quantities K, ψ, C, etc., are to be computed. No example of the way of effecting this computation is necessary. It is different, however, with the coefficients $b_n^{(i)}$, the computation of which may be greatly facilitated by a suitable arrangement of the work, as will be shown in a specimen of the computation. On page 247 is given a transcript from the computing book of all the figures made in computing a set of values of $b_n^{(i)}$ up to $i=20$, except the computation of $b_n^{(0)}$, $p_n^{(1)}$, and the last value of γ_i. The process is as follows:

The values of log λ_i are formed by adding to each $\mu^{(i)}$ the logarithm of sin ${}^2\chi$

$$\log \sin^2\chi = 9.9583004$$

and are written in the second column opposite the corresponding values of i.

The last value of log γ_i (in the computation log γ_{32}, but, in printing, log γ_{20}) is then written under the corresponding value of λ_i. The sum of the two logarithms is written opposite i in the third column and immediately under this sum its arithmetical complement, to serve as the argument in ZECH's table of subtraction logarithms. With this argument the value of log γ_{i-1} is taken from the table and written under log λ_{i-1}. This process is repeated upward until the value of γ_1 is reached.

To render easy the detection of an error the remainder of the work is done by the continual addition of differences. Since

$$\log p_i = \log \sin \chi + \log \theta_i + \log \gamma_i$$

we have

$$\triangle \log p_i = \triangle \log \theta_i + \triangle \log \gamma_i$$

The values of $\triangle$ log γ_i are formed by subtraction of the consecutive values of log γ_i and those of log θ_i by taking the differences of the Table of values of log θ_i. The remainder of the operation of forming log $b^{(i)}$ needs no explanation.

The above computation has to be made for each of a certain number of equidistant values of g, extending through the circle. In the case of Venus and Mars I have taken

$$g = 0^\circ;\ 15^\circ;\ 30^\circ;\ \ldots\ldots\ 345^\circ$$

thus dividing the circle into 24 parts.

The next step is, for each value of g and of i, to compute

$$b^{(i)} \cos i\,(Q-g) \text{ and } b^{(i)} \sin i\,(Q-g) \qquad (145)$$

where, in each case, the special numerical value of g, used in computing $b^{(i)}$ and Q, is to be used.

With the 24 special values of each of the quantities (145) its general expression in a trigonometric series, proceeding according to sines and cosines of g, is to be formed. I deem **Encke**'s method of forming this series to be the most convenient. For the case of $n=24$ the formulæ are as follows:

We put

$(m)=$numerical value of the function of $g=15^\circ\times m$ in (145)

We compute

$$\left.\begin{array}{ll} C^0=(0); & C^k=(k)+(24-k) \\ C^{12}=(12); & S^k=(k)-(24-k) \end{array}\right\} (k=1;\ 2;\ 3;\ \ldots\ 11)$$

$$C^6_+=C^6;\quad C^k_+=C^k+C^{12-k};\quad C_-=C^k-C^{12-k}\quad (k=0;\ 1;\ 2;\ 3;\ 4;\ 5)$$

$$S^6_+=S^6;\quad S^k_+=S_k+S^{12-k};\quad S^k_-=S^k-S^{12-k}\quad (k=1;\ 2;\ 3;\ 4;\ 5)$$

$$C^k_{++}=C^k_++C^{6-k}_+;\quad C^k_{+-}=C^k_+-C^{6-k}_+\quad (k=0;\ 1;\ 2)$$

$$C^k_{-+}=C^k_-+C^{6-k}_-;\quad C^k_{--}=C^k_--C^{6-k}_-\quad (k=1)$$

$$S^k_{++}=S^k_++S^{6-k}_+;\quad S^k_{+-}=S^k_+-S^{6-k}_+\quad (k=1)$$

$$S^k_{-+}=S^k_-+S^{6-k}_-;\quad S^k_{--}=S^k_--S^{6-k}_-\quad (k=1;\ 2)$$

$$a_0=C^1{}_{--}-C^3_-\qquad b_0=S^1_{+-}+S^3_+$$

$$a_1=C^0_-+\frac{1}{2}C^4_-\qquad b_1=\frac{1}{2}\sqrt{\frac{3}{2}}S^1{}_{++}$$

$$a_2=\frac{1}{2}\sqrt{3}\,C^2_-\qquad b_2=\sqrt{\frac{1}{2}}\left(\frac{1}{2}S^1{}_{+-}-S^3_+\right)$$

$$a_3=\frac{1}{2}\sqrt{\frac{3}{2}}C^1{}_{-+}\qquad b_3=\frac{1}{2}S^2_++S^6$$

$$a_4=\sqrt{\frac{1}{2}}\left(\frac{1}{2}C^1{}_{--}+C^3_-\right)\qquad b_4=\frac{1}{2}\sqrt{3}\,S^4_+$$

$$12\,(c_0+c_{12})=C^0_{++}+C^2_{++}$$

$$12\,(c_0-c_{12})=C^1{}_{++}+C^3_+$$

$$6\,(c_1+c_{11})=a_1+a_2\qquad 6\,(s_1+s_{11})=b_1-b_2$$

$$6\,(c_1-c_{11})=a_3+a_4\qquad 6\,(s_1-s_{11})=b_3+b_4$$

$$6\,(c_2+c_{10})=C^0_{+-}+\frac{1}{2}C^2_{+-}\qquad 6\,(s_2+s_{10})=\frac{1}{2}S^1{}_{-+}+S^3_-$$

$$6\,(c_2-c_{10})=\frac{1}{2}\sqrt{3}\,C^1{}_{+-}\qquad 6\,(s_2-s_{10})=\frac{1}{2}\sqrt{3}\,S^2_{-+}$$

$$6\,(c_3+c_9)=C^0_--C^4_-\qquad 6\,(s_3+s_9)=\sqrt{\frac{1}{2}}\,b_0$$

$$6\,(c_3-c_9)=\sqrt{\frac{1}{2}}\,a_0 \qquad 6\,(s_3-s_9)=S^2_+-S^6$$

$$6\,(c_4+c_8)=C^0_{++}-\frac{1}{2}\,C^2_{++} \qquad 6\,(s_4+s_8)=\frac{1}{2}\sqrt{3}\,S^1{}_{--}$$

$$6\,(c_4-c_8)=\frac{1}{2}\,C^1{}_{++}-C^3_+ \qquad 6\,(s_4-s_8)=\frac{1}{2}\sqrt{3}\,S^2_{--}$$

$$6\,(c_5+c_7)=a_1-a_2 \qquad 6\,(s_5+s_7)=b_1+b_2$$

$$6\,(c_5-c_7)=a_3-a_4 \qquad 6\,(s_5-s_7)=b_3-b_4$$

$$12\,c_6=C^0_{+-}-C^2_{+-} \qquad 12\,s_6=S^1{}_{-+}-S^3_-$$

It should be noted that the indices of C and S are not powers. The preceding formulæ may be controlled at pleasure by the general conditions

$$c_i=(0)+(1)\cos i.15^\circ+(2)\cos i.30^\circ+(3)\cos i.45^\circ+\ \ldots$$

$$s_i=\qquad (1)\sin i.15^\circ+(2)\sin i.30^\circ+(3)\sin i.45^\circ+\ \ldots$$

or by

$$c_0+c_1\cos n\,.\,15^\circ+c_2\cos 2\,n\,.\,15^\circ+c_3\cos 3\,n\,.\,15^\circ+\ \ldots$$
$$+s_1\sin n\,.\,15^\circ+s_2\sin 2\,n\,.\,15^\circ+s_3\sin 3\,n\,.\,15^\circ\ +\ \ldots\quad=(n)$$

The values of c_i and s_i obtained in this way from the special values of $b_n^{(i)}$ cos i Q′, and the accented coefficients c' and s' obtained from $b_n^{(i)}$ sin i Q′ for each value of i, being substituted in (141), give the required development of ρ_0^{-n} in the form (142).

The following form for the principal part of the computations was used:

	0	1	2	3	4	5	6
	(0)	(1)	(2)	(3)	(4)	(5)	
		(23)	(22)	(21)	(20)	(19)	
		(11)	(10)	(9)	(8)	(7)	(6)
	(12)	(13)	(14)	(15)	(16)	(17)	(18)
C	C^0	C^1	C^2	C^3	C^4	C^5	C^6
	C^{12}	C^{11}	C^{10}	C^9	C^8	C^7	
S		S^1	S^2	S^3	S^4	S^5	S^6
		S^{11}	S^{10}	S^9	S^8	S^7	
C_+	C^0_+	C^1_+	C^2_+	C^3_+			
	C^6_+	C^5_+	C^4_+				
			$\log C^2_-$				
C_-	C^0_-	C^1_-	C^2_-	C^3_-			
	$\frac{1}{2}C^4_-$	C^5_-	C^4_-	$\frac{1}{2}C^1_{--}$			
			C^0_-				
C_{++}	C^0_{++}	C^1_{++}	C^2_{++}	C^3_+			
	$\frac{1}{2}C^2_{++}$	C^3_+	C^0_{++}	$\frac{1}{2}C^1_{++}$			
C_{+-}	C^0_{+-}	C^1_{+-}	C^2_{+-}				
	$\frac{1}{2}C^2_{+-}$	$\log C^1_{+-}$	C^0_{+-}				
C_{-+}		C^1_{-+}					
		$\log C^1_{-+}$					
C_{--}		C^1_{--}					
		C^3_-	S^6				
S_+		S^1_+	S^2_+	S^3_+			
	S^6	S^5_+	S^4_+				
	$\frac{1}{2}S^2_+$		$\log S^4_+$				
S_-		S^1_-	S^2_-	S^3_-			
		S^5_-	S^4_-	$\frac{1}{2}S^1_{-+}$			
S_{++}		S^1_{++}	$\log S^1_{++}$				
S_{+-}		S^1_{+-}					
		S^3_+					
S_{-+}		S^1_{-+}	S^2_{-+}				
		S^3_-					
S_{--}		S^1_{--}	S^2_{--}				

In presenting the following numerical data and results the writer has been guided by the consideration that future investigators may desire to repeat his work with a higher degree of numerical precision than that which he has himself reached. In this connection it may be remarked that complete numerical consistency in the use of the adopted elements seems to be necessary. For example, serious error might result in the terms of very high order if one value of the eccentricity were used in one part of the computation and another value in another, and this, although neither of the two values should be seriously in error. It is also to be remembered that small changes in the elements, especially the positions of the perihelia and nodes, may make considerable changes in the coefficients of the terms in question. The epoch 1800 was therefore chosen as that to which the developments should refer, because this was fairly near to the mean of the time through which observations of the Moon's position were available. If the elements for 1900 were used, it is possible that slightly different results might be found for the inequalities of very long period, as, for instance, HANSEN'S inequality due to the action of Venus.

In the actual computations the true anomalies and radii vectores of the Earth and Mars for different values of their mean anomaly enter as fundamental numerical data in the computation. These anomalies and radii vectores were taken for 1800 out of LE VERRIER'S tables. On the other hand, where the eccentricities enter into the formulæ, they were reduced to 1800 with LE VERRIER'S data. It is not impossible that in this way numerical inconsistencies, amounting to a large fraction of a second of arc, have crept into the computation, especially as account was not taken of the minute error in the secular variation of the Earth's radius vector which exists in LE VERRIER'S tables of the Sun. It must not, therefore, be understood that the whole computation is absolutely consistent with the following numerical elements which were intended to form their basis. I shall therefore present such derived quantities as will enable the general consistency of the work to be most easily tested.

In the notation of § 17 it will be seen that the position of the planet whose elements are accented remains indeterminate, while the computations are made for special values of the mean anomaly which is unaccented. Now, in the respective cases of Venus-Earth and Earth-Mars, it is most convenient to let the outer planet play the part of the unaccented one, since its eccentricity is in each case the largest.

The values of the fundamental elements actually used in the computations were as follows:

Adopted quantities for 1800.

	Venus and Earth.			Earth and Mars.		
	°	′	″	°	′	″
π	99	29	56	332	22	42
π'	128	46	1	99	29	56
θ	254	52	28	48	0	33
γ	3	23	32.5	1	51	3·5
$\pi-\theta=\omega$	204	37	28	284	22	9
$\pi'-\theta=\omega'$	233	53	33	51	29	23
$\omega-K_1$	330	41	3	232	53	38.3
$\omega-K_2$	330	46	48	232	51	53·7
e'	0.0068703			0.016792		
$\log k_1$	9·9997360			9·999861		
$\log k_2$	9·9995030			9·999912		
$\log 2\,e'\,a'^2$	7.856682			8.52614		
$\log s$	5·3926			6.4502		

The data and principal results of the computations made by the preceding method are given in the pages next following with some fullness, not only on account of their bearing upon the problem immediately in hand, but of their possible usefulness in the theories of Venus, the Earth, and Mars. In the Hansenian method of developing the perturbative function, the inverse first, third, and fifth powers of the distances of the two planets must first be developed as the basis of the whole computation. From this development the function itself and its derivatives is derived by the process called by **Hansen** "mechanical multiplication;" that is to say, the factors by which the inverse powers are multiplied are expressed as functions of the mean anomaly of one or the other of the planets, developable in sines or cosines of multiples of that quantity. The coefficients in this development are pure numbers, so that only a numerical multiplication is necessary.

We first have to compute the values of the coefficients which enter into the expressions (100), (116), (119), and (126). This computation is made by the formulæ (124) and (125).

For the separate computations the circle has been divided into 24 parts; a number greater is rigorously necessary, but advisable in order that the general consistency of the results may afford a check upon the accuracy of the computations.

VENUS AND THE EARTH.

Values of the principal quantities for each 15° of the Earth's mean anomaly.

g	log K sin ψ	log K cos ψ	C	Q	log q	log $b_3^{(0)}$	log $b_5^{(0)}$	log sin χ
°				° ′ ″				
0	9.8426013*n*	0.0959176	1.4984309	330 50 5.3	0.1547869	1.0393590	2.0106083	9.9791502
15	9.5307811*n*	0.1424754	1.5005595	346 15 34.0	0.1550778	1.0321281	1.9961461	9.9788246
30	8.6167445	0.1555235	1.5042569	1 39 24.0	0.1556971	1.0219812	1.9759630	9.9783751
45	9.6225171	0.1372497	1.5092620	16 59 52.6	0.1566407	1.0105490	1.9533528	9.9778760
60	9.8851684	0.0850808	1.5152267	32 15 22.8	0.1578726	0.9992563	1.9311747	9.9773950
75	0.0263142	9.9897552	1.5217372	47 24 35.2	0.1593188	0.9890811	1.9113730	9.9769792
90	0.1085707	9.8261023	1.5283468	62 26 37.7	0.1608716	0.9805013	1.8948515	9.9766495
105	0.1517266	9.5027179	1.5346068	77 21 13.2	0.1624003	0.9735970	1.8817050	9.9764032
120	0.1634551	8.7370093*n*	1.5400942	92 8 42.6	0.1637673	0.9681470	1.8714128	9.9762200
135	0.1458188	9.6266576*n*	1.5444443	106 50 2.9	0.1648478	0.9638634	1.8633261	9.9760756
150	0.0965588	9.8829568*n*	1.5473683	121 26 41.9	0.1655454	0.9605464	1.8569903	9.9759517
165	0.0075048	0.0228020*n*	1.5486731	136 0 28.2	0.1658028	0.9581767	1.8523352	9.9758430
180	+9.8571886	0.1055524*n*	1.5482730	150 33 21.5	0.1656082	0.9569734	1.8497980	9.9757607
195	9.5745021	0.1501937*n*	1.5461945	165 7 21.1	0.1649943	0.9573378	1.8502075	9.9757301
210	7.8244550	0.1640344*n*	1.5425743	179 44 16.2	0.1640312	0.9597802	1.8546328	9.9757851
225	9.5592682*n*	0.1489068*n*	1.5376514	194 25 38.0	0.1628151	0.9647424	1.8640192	9.9759572
240	9.8498682*n*	0.1023996*n*	1.5317533	209 12 33.3	0.1614556	0.9724792	1.8789404	9.9762668
255	0.0025697*n*	0.0163044*n*	1.5252760	224 5 42.6	0.1600604	0.9828938	1.8992569	9.9767119
270	0.0921850*n*	9.8694460*n*	1.5186557	239 5 20.2	0.1587229	0.9954332	1.9238996	9.9772636
285	0.1407563*n*	9.5928653*n*	1.5123460	254 11 16.3	0.1575166	1.0090239	1.9507358	9.9778654
300	0.1564598*n*	8.1880642*n*	1.5067814	269 23 1.8	0.1564928	1.0221600	1.9767483	9.9784426
315	0.1412940*n*	9.5590788	1.5023484	284 39 51.5	0.1556843	1.0330654	1.9983638	9.9789136
330	0.0925806*n*	9.8542650	1.4993584	300 0 47.2	0.1551152	1.0401244	2.0123302	9.9792098
345	0.0011484*n*	0.0073922	1.4980207	315 24 38.9	0.1548076	1.0422776	2.0165188	9.9792898

MARS AND THE EARTH.

Values of the principal quantities for each 15° of the mean anomaly of Mars.

g	log K sin ψ	log K cos ψ	C	log q	Q	log $b_{3}^{(0)}$	log q	log sin χ
°					° ′ ″			
0	0.342858*n*	0.212979*n*	2.881190	0.438017	233 26 53.9	0.599924	6.0122	9.978445
15	0.418731*n*	9.939190*n*	2.910317	0.441407	251 39 51.6	0.575359	6.0088	9.977466
30	0.448288*n*	8.384265*n*	2.971771	0.448350	269 30 20.5	0.525201	6.0018	9.975335
45	0.439091*n*	9.917232	3.060116	0.457926	286 43 58.9	0.457934	5.9923	9.972189
60	0.391572*n*	0.207560	3.167944	0.469034	303 12 14.5	0.382842	5.9812	9.968257
75	0.298636*n*	0.357601	3.286834	0.480623	318 52 19.0	0.307729	5.9696	9.963845
90	0.137266*n*	0.444640	3.408158	0.491815	333 45 49.5	0.238109	5.9581	9.959295
105	9.821323*n*	0.492325	3.523778	0.501949	347 57 24.5	0.177420	5.9483	9.954940
120	8.944271	0.510437	3.626471	0.510555	1 33 20.6	0.127466	5.9396	9.951070
135	9.921101	0.502957	3.710223	0.517327	14 40 44.9	0.089206	5.9329	9.947927
150	0.185703	0.470233	3.770379	0.522066	27 27 0.6	0.062981	5.9281	9.945680
165	0.332616	0.408997	3.803729	0.524655	39 59 34.1	0.048896	5.9255	9.944445
180	0.424058	0.310220	3.808538	0.525041	52 25 48.2	0.047016	6.9252	9.944282
195	0.480036	0.151066	3.784559	0.523210	64 53 5.7	0.057224	5.9270	9.945195
210	0.508707	9.855220	3.733036	0.519199	77 28 52.3	0.079588	5.9310	9.947137
225	0.513048	8.291540*n*	3.656695	0.513098	90 20 38.3	0.113981	5.9371	9.950010
240	0.492674	9.876442*n*	3.559711	0.505068	103 36 0.2	0.160083	5.9451	9.953654
255	0.443756	0.158005*n*	3.447645	0.495379	117 22 30.6	0.217243	5.9548	9.957857
270	0.356911	0.308213*n*	3.327367	0.484444	131 47 19.0	0.283875	5.9658	9.962343
285	0.209610	0.396210*n*	3.206725	0.472863	146 56 28.9	0.357230	5.9773	9.966801
300	9.929867	0.441856*n*	3.094241	0.461452	162 53 52.3	0.432631	5.9887	9.970898
315	8.223455*n*	0.451265*n*	2.998513	0.451227	179 39 39.0	0.503153	5.9990	9.974321
330	9.912794*n*	0.423601*n*	2.927385	0.443299	197 8 46.0	0.560062	6.0069	9.976819
345	0.199031*n*	0.351198*n*	2.887057	0.438679	215 10 3.2	0.594434	6.0115	9.978223

Table of the valves of log $\theta_n^{(i)}$ *and log* $\mu_n^{(i)}$ *for the computation of* $b_n^{(i)}$.

i	$n=3$			$n=5$		
	log $\theta^{(i)}$	Δ log $\theta^{(i)}$	log $\mu^{(i)}$	log $\theta^{(i)}$	Δ log $\theta^{(i)}$	log $\mu^{(i)}$
1	9.8750613	791813	. . . . . .	0.0969100	1549020	. . . .
2	9.7958800	299632	9.1938200	9.9420080	669467	9.3399480n
3	9.7659168	. . . . .	9.3399481	9.8750613	377886	8.9719713
4	9.7501225	157943	9.3699113	9.8372727	243593	9.2352128
5	9.7403627	97598	9.3813407	9.8129134	170333	9.3077634
6	9.7337322	66305	9.3869447	9.7958801	125892	9.3399481
7	9.7289333	47989	9.3901148	9.7832909	96873	9.3573222
8	9.7252989	36344	9.3920843	9.7736036	76868	9.3678383
9	9.7224511	28478	9.3933924	9.7659168	62490	9.3747102
10	9.7201593	22918	9.3943057	9.7596678	51805	9.3794566
11	9.7182751	18842	9.3949687	9.7544873	43647	9.3828762
12	9.7166988	15763	9.3954654	9.7501226	37279	9.3854235
13	9.7153605	13383	9.3958471	9.7463947	32210	9.3873728
14	9.7142100	11505	9.3961467	9.7431737	28111	9.3888982
15	9.7132104	9996	9.3963862	9.7403626	24746	9.3901146
16	9.7123339	8765	9.3965806	9.7378880	21952	9.3911005
17	9.7115591	7748	9 3967408	9.7356928	19607	9.3919108
18	9.7108692	6899	9.3968742	9.7337321	17618	9.3925849
19	9.7102510	6182	9.3969865	9.7319703	15918	9.3931517
20	9.7096939	5571	9.3970820	9.7303785	14453	9.3936329
21	9.7091892	5047	9.3971638	9.7289332	. . . . . .	9.3940449
22	9.7087298	4594	9.3972344	9.7276152	13180	9.3944005
23	9.7083101	4197	9.3972959	9.7264083	12069	9.3947095
24	9.7079249	3852	9.3973496	9.7252990	11093	9.3949797
25	9.7075702	3547	9.3973969	9.7242759	10231	9.3952172
26	9.7072426	3276	9.3974387	9.7233294	9465	9.3954273
27	9.7069389	3037	9.3974758	9.7224511	8783	9.3956139
28	9.7066569	2820	9.3975090	9.7216340	8171	9.3957804
29	9.7063940	2629	9.3975387	9.7208718	7622	9.3959297
30	9.7061486	2454	9.3975655	9.7201593	7125	9.3960640
31	9.7059188	2298	9.3975896	9.7194917	6676	9.3961854
32	9.7057034	2154	9.3976115	9.7188648	6269	9.3962948

The following is a complete copy of the computation of $b_5^{(i)}$ for Venus when $g=0°$, showing every figure that was made, as explained on pages 233–234:

VENUS: *Computation of $b_5^{(i)}$ for g (Earth)=0°.*

	log λ_i log γ_i	log $\lambda_i \gamma_i$	Δ log γ_i Δ log p_i	log $b_5^{(i)}$ log p_i
	. . . .	. . . .	. . . .	2.0106083
1	. . . .	. . . .	. . . .	9.9887235
	9.9126633	. . . .	—1369007	1.9993318
2	—9.2982484	9.3478124	- 180013	9.9707222
	0.0495640	0.6521876	529996	1.9700540
3	+8.9302717	9.0328353	— 139471	9.9567751
	0.1025636	0.9671647	268591	1.9268291
4	9.1935132	9.3229359	— 109295	9.9458456
	0.1294227	0.6770641	156343	1.8726747
5	9.2660638	9.4111208	— 87250	9.9371206
	0.1450570	0.5888792	99373	1.8097953
6	9.2982485	9.4532428	— 70960	9.9300246
	0.1549943	0.5467572	67197	1.7398199
7	9.3156226	9.4773366	— 58695	9.9241551
	0.1617140	0.5226634	47596	1.6639750
8	9.3261387	9.4926123	— 49277	9.9192274
	0.1664736	0.5073877	34956	1.5832024
9	9.3330106	9.5029798	— 41912	9.9150362
	0.1699692	0.4970202	26434	1.4982386
10	9.3377570	9.5103696	-- 36056	9.9114306
	0.1726126	0.4896304	20475	1.4096692
11	9.3411766	9.5158366	— 31330	9.9082976
	0.1746600	0.4841634	16182	1.3179668
12	9.3437239	9.5200022	— 27465	9.9055511
	0.1762783	0.4799978	13010	1.2235179
13	9.3456732	9.5232524	— 24269	9.9031242
	0.1775792	0.4767476	10617	1.1266421
14	9.3471986	9.5258395	— 21593	9.9009649
	0.1786409	0.4741605	8777	1.0276070
15	9.3484150	9.5279346	— 19334	9.8990315
	0.1795186	0.4720654	7339	0.9266385
16	9.3494009	9.5296534	— 17407	9.8972908
	0.1802525	0.4703466	6198	0.8239293
17	9.3502112	9.5310835	— 15754	9.8957154
	0.1808723	0.4689165	5282	0.7196447
18	9.3508853	9.5322858	— 14325	9.8942829
	0.1814005	0.4677142	4537	0.6139276
19	9.3514521	9.5333063	— 13081	9.8929748
	0.1818542	0.4666937	3925	0.5069024
20	9.3519333	9.5341801	— 11993	9.8917755
	0.1822468	0.4658199	3424	0.3986779

VENUS AND THE EARTH.

Numerical development of 24 times the inverse third power of the distance of Venus from the Earth in terms of the mean anomalies of the two planets in their elliptic orbits for 1900.

$$\frac{1}{\rho^3}=\Sigma\left\{a\cos\left(i'g'+i_4g_4\right)+b\sin\left(i'g'+i_4g_4\right)\right\}$$

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
0 0	+ 118.3612	. . . . . .	− 4 − 1	− 0.0025	+ 0.0028
+ 1	+ 5.7884	− 5.3696	0	+ .0036	+ 0.0298
2	− 0.5389	− 1.9642	1	+ .2163	+ 0.1177
3	− 0.1207	− .0838	2	+ 2.1331	− 0.6524
4	− 0.0190	+ .0090	3	+ 3.4226	− 13.2752
5	− 0.0011	+ .0018	4	− 50.2777	− 98.3635
− 1 − 4	+ .0028	− .0014	5	− 1.1882	− .7570
− 3	− .0232	+ .0018	6	− 0.8552	+ .8260
− 2	− .0787	+ .1582	7	− 0.0124	+ .0545
− 1	+ .5543	+ 2.1122	8	+ 0.0098	+ .0062
0	+ 10.5927	+ 2.3432	9	+ 0.0005	+ .0005
+ 1	+ 183.0618	− 102.6121	− 5 + 1	+ 0.0195	+ 0.0248
2	+ 1.7962	− 5.1687	2	+ .2559	− .0058
3	− 1.3062	− 1.2980	3	+ 1.4302	− 1.5819
4	− .1188	− .0134	4	− 3.3224	− 12.3912
5	− .0102	+ .0154	5	− 71.4518	− 47.5899
6	+ .0002	− .0003	6	− 0.7918	− .1739
− 2 − 3	− .0025	− .0006	7	− 0.2824	+ .9536
− 2	− .0209	+ .0145	8	+ .0106	+ .0404
− 1	+ .0084	+ .2047	9	+ .0102	+ .0005
0	+ 1.6027	+ 1.5935	10	+ .0005	− .0004
+ 1	+ 12.5252	− 3.6702	− 6 + 1	+ 0.0009	+ 0.0034
2	+ 90.9943	− 148.7331	2	+ .0302	+ 0.0125
3	− .4977	− 3.6032	3	+ .2204	− .1339
4	− 1.5665	− 0.4238	4	+ .3934	− 1.9542
5	− .0875	+ 0.0352	5	− 8.0905	− 8.2552
6	− .0010	+ 0.0160	6	− 65.8513	− 5.0742
7	+ .0008	− 0.0035	7	− .4463	+ .0008
− 3 − 2	− .0027	+ 0.0005	8	+ .1845	+ .8016
− 1	− .0112	+ 0.0255	9	+ .0206	+ .0234
0	+ .1208	+ 0.1948	10	+ .0075	− .0042
+ 1	+ 2.2085	+ 0.5489	11	+ .0006	+ .0002
2	+ 9.6563	− 9.9992	− 7 + 2	+ .0034	+ 0.0030
3	− 5.3594	− 140.0914	3	+ .0325	− 0.0052
4	− 1.2859	− 1.9297	4	+ .1215	− .2214
5	− 1.3566	+ 0.3500	5	− .6007	− 1.7245
0	− 0.0474	+ 0.0562	6	− 9.7205	− 2.8233
7	+ 0.0064	+ 0.0122	7	− 45.7518	+ 21.1876
8	+ 0.0011	+ .0006	8	− .2665	− .0015

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse third power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
− 7 + 9	+ .4540	+ .4971	− 11 + 13	+ .0963	− .2625
10	+ .0211	+ .0085	14	+ .0016	− .0059
11	+ .0033	− .0053	15	− .0034	− .0006
12	+ .0004	− .0004	16	− .0003	0000
− 8 + 3	+ .0042	+ 0.0005	− 12 + 7	− 0.0014	+ 0.0008
4	+ .0255	− .0206	8	− .0250	− 0.0085
5	− .0055	− .2418	9	− .1452	+ .0888
6	− 1.2516	− 1.0642	10	− .1604	+ .9208
7	− 8.4424	+ 1.9735	11	+ 2.6107	+ 3.0627
8	− 22.4300	+ 30.9883	12	+ 12.0025	+ 1.8825
9	− .2175	− .0212	13	− .0779	+ 0.1720
10	+ .5188	+ .1705	14	− .0350	− .2180
11	+ .0155	− .0011	15	− .0004	− .0042
12	− .0002	− .0062	16	− .0024	+ .0009
13	− .0005	− .0007	17	− .0002	0000
− 9 + 4	− 0.0006	− 0.0015	− 13 + 8	− 0.0031	− 0.0024
5	+ .0112	− .0296	9	− .0242	+ .0045
6	− .1184	− .1940	10	− .0722	+ .1313
7	− 1.4322	− .2547	11	+ .2694	+ .7423
8	− 5.3796	+ 4.9292	12	+ 3.0353	+ 1.1206
9	− 3.3436	+ 28.6854	13	+ 8.4873	− 3.1441
10	− .2202	+ .0054	14	+ .0056	+ .1654
11	+ .4307	− .0860	15	− .1084	− .1357
12	+ .0105	− .0048	16	− .0011	− .0026
13	− .0024	− .0045	17	− .0012	+ .0016
14	0000	+ .0001	18	0000	+ .0002
− 10 + 5	+ 0.0025	− 0.0032	− 14 + 9	− 0.0026	− 0.0002
6	− .0053	− .0302	10	− .0170	+ .0143
7	− .1835	− .1005	11	+ .0020	+ .1315
8	− 1.1930	+ .4365	12	+ .5045	+ .4256
9	− 1.8885	+ 5.7498	13	+ 2.5500	− .4050
10	+ 8.3438	+ 20.0448	14	+ 4.3630	− 5.1199
11	− .2082	+ .0699	15	+ .0685	+ .1226
12	+ .2669	− .2289	16	− .1263	− .0514
13	+ .0053	− .0062	17	− .0012	− .0016
14	− .0034	− .0024	18	− .0002	+ .0017
15	− .0003	+ .0012	19	− .0003	− .0004
− 11 + 6	+ 0.0007	− 0.0021	− 15 + 10	− 0.0028	+ 0.0012
7	− 0.0183	− .0217	11	− .0065	+ .0179
8	− .1899	+ .0042	12	+ .0590	+ .0971
9	− .7018	+ .8408	13	+ .5388	+ .0992
10	+ .9582	+ 4.8665	14	+ 1.6070	− 1.2697
11	+ 12.7915	+ 10.0493	15	+ .9693	− 4.8945
12	− .1571	+ .1368	16	+ .0989	+ .0652

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse third power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
−15+17	− .1052	+ .0133	−19+21	+ .0220	+ .0312
18	− .0012	− .0008	22	− .0010	+ .0010
19	+ .0003	+ .0015	23	+ .0006	− .0002
20	− .0001	− .0002	24	− .0008	− .0002
−16+11	− 0.0019	− 0.0014	−20+15	+ 0.0017	+ 0.0013
12	+ .0038	+ .0173	16	+ .0076	− .0069
13	+ .0850	+ .0475	17	0000	− .0498
14	+ .4264	− .1495	18	− .1454	− .1314
15	+ .6158	− 1.4937	19	− .5933	+ .0490
16	− 1.1430	− 3.5132	20	− .7741	+ .7724
17	+ .0966	+ .0094	21	− .0135	− .0381
18	− .06638	+ .04913	22	+ .0257	+ .0135
19	− .0012	− .0002	23	− .0006	+ .0006
20	+ .0009	+ .0005	24	+ .0005	− .0010
21	+ .0002	+ .0004	25	+ .0002	0000
−17+12	− 0.0006	+ 0.0027	−21+16	+ 0.0015	− 0.0006
13	+ .0103	+ .0116	17	+ .0030	− .0082
14	+ .0835	− .0010	18	− .0203	− .0355
15	+ .2425	− .2790	19	− .1534	− .0342
16	− .1522	− 1.2574	20	− .3810	+ .2566
17	− 2.0054	− 1.8541	21	− .2198	+ .7735
18	+ .0732	− 0300	22	− .0235	− .0220
19	− .0264	+ .0584	23	+ .0225	− .0005
20	− .0010	0000	24	− .0002	+ .0014
21	+ .0010	− .0003	25	− .0001	− .0004
22	− .0003	+ .0009	26	0000	− .0004
−18+13	+ 0.0013	+ 0.0025	−22+17	+ 0.0007	− 0.0011
14	+ .0130	+ .0045	18	− .0014	− .0072
15	+ .0608	− .0358	19	− .0285	− .0174
16	+ .0592	− .2955	20	− .1200	+ .0367
17	− .5833	− .7981	21	− .1603	+ .3160
18	− 1.9597	− .4708	22	+ .1359	+ .5745
19	+ .0408	− .0494	23	− .0235	− .0062
20	+ .0042	+ .0488	24	+ .0140	− .0096
21	− .0005	+ .0004	25	+ .0002	+ .0003
22	+ .0006	+ .0001	26	− .0002	− .0002
23	− .0001	0000	27	− .0003	− .0002
−19+14	+ 0.0016	+ 0.0013	−23+18	+ 0.0004	− 0.0014
15	+ .0118	− 0.0020	19	− .0041	− .0050
16	+ .0298	− .0509	20	− .0276	− .0005
17	− .0758	− .2314	21	− .0690	+ .0725
18	− .6988	− .3196	22	+ .0110	+ .2710
19	− 1.4302	+ .4021	23	+ .2931	+ .3202
20	+ .0095	− .0500	24	− .0180	+ .0040

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse third power, etc.—Continued.

i' i_4	24a	24b
−23+25	+ .0060	− .0116
26	+ .0002	.0000
27	− .0002	- .0002
−24+19	− 0.0001	− 0.0010
20	− .0048	− .0018
21	− .0198	+ .0106
22	− .0201	+ .0762
23	+ .1092	+ .1768
24	+ .3016	+ .1000
25	− .0109	+ .0096
26	.0000	− .0102
27	+ .0002	.0000
−25+20	− 0.0009	− 0.0008
21	− .0044	− .0013
22	− .0101	+ .0150
23	+ .0168	+ .0610
24	+ .1382	+ .0765
25	+ .2283	− .0440
26	− .0036	+ .0108
27	− .0037	− .0065
−26+21	− 0.0008	− 0.0004
22	− .0034	+ .0017
23	− .0007	+ .0148
24	+ .0352	+ .0348
25	+ .1205	− .0004
−26+26	+ .1301	− .1101
27	+ .0018	+ .0086
−27+22	− 0.0009	− 0.0010
23	− .0012	+ .0028
24	+ .0056	+ .0116
25	+ .0369	+ .0105
26	+ .0796	− .0454
27	+ .0438	− .1170
−28+23	0.0000	+ 0.0003
24	− .0002	+ .0021
25	+ .0080	+ .0046
26	+ .0294	− .0077
27	+ .0362	− .0600
−29+24	+ 0.0004	0.0000
25	+ .0013	+ .0015
26	+ .0078	+ .0008
27	+ .0172	− .0160
−30+25	. . . . . .	. . . . . .
26	. . . . . .	. . . . . .
27	. . . . . .	. . . . . .
−31+26	. . . . . .	. . . . . .
27	. . . . . .	. . . . . .
−32+27	. . . . . .	. . . . . .

VENUS AND THE EARTH.

Numerical development of 24 times the inverse fifth power of the distance of Venus from the Earth in terms of the mean anomalies of the two planets in their elliptic orbits for 1800.

$$\frac{1}{\rho^5}=\Sigma\left\{a\cos(i'g'+i_4g_4)+b\sin(i'g'+i_4g_4)\right\}$$

i'	i_4	$24a$	$24b$
0	0	+1006.4170	. . . .
	+1	+119.0931	−106.3205
	2	−8.8087	−38.0411
	3	−3.4431	−2.3838
	4	−0.5932	+0.2470
	5	+.0854	+.0702
	6	+.0045	−.0039
	7	+.0006	−.0090
−1	−6	+.0024	−.0010
	−5	+.0129	−.0071
	−4	−.0673	−.0532
	−3	−.7002	+.0931
	−2	−2.0962	+4.2720
	−1	+12.1686	+39.6173
	0	+181.1343	+40.4850
	+1	+1706.5824	−956.5143
	2	+43.7904	−124.9798
	3	−24.6545	−26.4264
	4	−3.5975	−0.3571
	5	−0.3468	+0.4573
	6	+.0081	+.0650
	7	+.0054	+.0028
	8	−.0008	−.0030
−2	−5	+.0067	+.0009
	−4	−.0050	−.0063
	−3	−.0886	−.0163
	−2	−.6092	+.4736
	−1	+.2904	+5.2895
	0	+31.7583	+29.2754
	+1	+197.4674	−59.2818
	2	+948.9104	−1550.8436
	3	−17.4126	−105.1862
	4	−31.4258	−9.6859
	5	−2.8180	+1.2074
	6	−.0606	+.5064
	7	+.0336	+.0390
	8	+.0051	−.0004
	9	−.0021	−.0002
−3	−4	+.0039	−0.0004
	−3	−.0083	−.0112
−3	−2	−.0981	+.0387
	−1	−.3112	+.7738
	0	+3.1062	+4.8577
	+1	+43.1818	+9.4122
	2	+151.6440	−157.4512
	3	+62.5029	−1631.3784
	4	−51.3790	−66.0423
	5	−28.7837	+6.4600
	6	−1.5760	+2.0155
	7	+.1773	+0.4129
	8	+.0441	+.0199
	9	+.0034	−.0010
	+10	−.0005	−.0008
−4	−3	+.0036	+.0001
	−2	−.0104	−.0124
	−1	−.0682	−.1264
	0	+.1310	+.8778
	+1	+5.4265	+2.8633
	2	+42.1948	−13.8210
	3	+55.8009	−215.1555
	4	−649.5585	−1270.9491
	5	−58.7327	−25.9984
	6	−19.2802	+17.5802
	7	−.3446	+2.0762
	8	+.3152	+0.2342
	9	+.0411	−0.0018
	10	+.0023	−.0022
	+11	−.0007	+.0009
−5	−2	+.0024	−.0018
	−1	−.0078	+.0076
	0	−.0205	+.1192
	+1	+.5847	+.7284
	2	+6.3901	−.2083
	3	+28.9857	−32.8389
	4	−55.8416	−210.7800
	5	−1016.7720	−677.2520
	6	−48.1612	+3.4306
	7	−7.0478	+21.6610
	8	+0.5576	+1.6135
	9	+.3400	+0.0402

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse fifth power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
− 5+10	+ .0278	− .0199	− 8+12	+ 0.0256	− 0.2130
11	− .0001	− .0030	13	− 0.0092	− 0.0160
12	− .0013	+ .0006	14	− 0.0024	+ 0.0003
− 6− 1	.0000	. . . .	15	+ 0.0001	+ 0.0006
0	− .0002	+ 0.0003	− 9+ 2	− 0.0044	− 0.0018
+ 1	+ .0460	− 0.0898	3	+ 0.0143	+ 0.0132
2	+ .9020	+ .3424	4	+ 0.1349	− 0.0494
3	+ 5.5517	− 3.4215	5	+ 0.3370	− 0.9244
4	+ 8.1177	− 41.6650	6	− 3.2152	− 5.2920
5	− 144.7228	− 148.7746	7	− 34.5082	− 6.2933
6	−1024.8068	− 78.9782	8	− 115.3828	+ 104.7718
7	− 30.3002	+ 18.7156	9	− 65.5964	+ 563.2880
8	+ 3.9405	+ 19.2787	10	+ 4.6548	+ 11.8749
9	+ 1.0080	+ 0.9265	11	+ 11.7690	− 2.0918
10	+ 0.2701	− .1169	12	+ 0.4940	− 0.4355
11	+ .0138	− .0262	13	− 0.0715	− 0.1671
12	− .0014	− .0028	14	− 0.0133	− 0.0080
13	− .0001	+ .0003	15	− 0.0016	+ 0.0012
− 7 0	0.0000	0.0000	16	+ 0.0009	+ 0.0001
+ 1	+ 0.0024	+ 0.0120	−10+ 3	− 0.0034	− 0.0011
2	+ 0.1040	+ 0.0844	4	+ 0.0185	− 0.0022
3	+ 0.9728	− 0.1614	5	− 0.0078	− 0.1103
4	+ 3.1142	− 5.7330	6	− 0.1714	− 0.9483
5	− 13.3523	− 38.1902	7	− 5.1623	− 2.8388
6	− 184.6949	− 54.4921	8	− 30.1037	+ 10.8131
7	− 773.6470	+ 358.3035	9	− 43.2380	+ 129.7126
8	− 13.3186	+ 22.0683	10	+ 175.3675	+ 421.1381
9	+ 11.0498	+ 12.6665	11	+ 6.2918	+ 6.0300
10	+ 1.0504	+ 0.2704	12	+ 7.6830	− 6.3400
11	+ 0.1498	− 0.2015	13	+ 0.1820	− 0.4718
12	− 0.0001	− 0.0235	14	− 0.1218	− 0.0920
13	− 0.0022	− 0.0012	15	− 0.0126	− 0.0005
14	− 0.0001	+ 0.0007	16	− 0.0005	+ 0.0016
− 8+ 1	+ 0.0010	+ 0.0066	17	+ 0.0013	+ 0.0007
2	+ 0.0080	− 0.0006	−11+ 4	− 0.0013	+ 0.0024
3	+ 0.1435	+ 0.0258	5	− 0.0006	− 0.0108
4	+ 0.7643	− 0.6336	6	+ 0.0236	− 0.1406
5	− 0.1532	− 6.4062	7	− 0.6030	− 0.7098
6	− 28.8606	− 24.6526	8	− 5.5253	+ 0.1065
7	− 170.4302	+ 39.0604	9	− 18.6115	+ 21.9767
8	− 409.7310	+ 566.1468	10	+ 22.3335	+ 116.1812
9	− 1.4276	+ 18.2510	11	+ 286.4366	+ 224.9622
10	+ 13.4256	+ 4.7282	12	+ 5.4702	+ 2.0475
11	+ 0.8253	− 0.2005	13	+ 2.9600	− 7.6945

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse fifth power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
−11+14	− 0.0420	− 0.3824	−14+18	− 0.0102	+ 0.0723
15	− 0.1272	− 0.0159	19	+ 0.0015	+ 0.0047
16	− 0.0090	+ 0.0040	20	+ 0.0006	− 0.0006
17	+ 0.0001	+ 0.0013	21	− 0.0002	+ 0.0013
18	− 0.0017	+ 0.0002	−15+ 8	+ 0.0014	+ 0.0001
−12+ 5	+ 0.0098	+ 0.0024	9	− 0.0166	− 0.0064
6	+ 0.0070	− 0.0064	10	− 0.1158	+ 0.0439
7	− 0.0489	− 0.1380	11	− 0.2402	+ 0.6828
8	− 0.8374	− 0.2978	12	+ 1.9330	+ 3.2702
9	− 4.3874	+ 2.6358	13	+ 16.7141	+ 3.2108
10	− 4.5948	+ 25.2182	14	+ 46.8696	− 36.7282
11	+ 65.2679	+ 77.2590	15	+ 27.0458	− 136.8284
12	+ 285.3858	+ 44.6933	16	+ 0.9186	− 0.8614
13	+ 3.8562	− 0.0322	17	− 3.6249	+ 0.4142
14	− 0.9870	− 6.7076	18	− 0.1100	+ 0.0450
15	− 0.1595	− 0.2420	19	+ 0.0232	+ 0.0555
16	− 0.0983	+ 0.0408	20	+ 0.0032	+ 0.0031
17	− 0.0047	+ 0.0070	21	+ 0.0019	− 0.0003
18	+ 0.0006	+ 0.0008	22	− 0.0004	0.0000
19	+ 0.0001	− 0.0007	−16+ 9	+ 0.0010	+ 0.0007
−13+ 6	+ 0.0006	− 0.0016	10	− 0.0169	+ 0.0005
7	− 0.0012	− 0.0180	11	− 0.0735	+ 0.0906
8	− 0.1048	− 0.0859	12	+ 0.1229	+ 0.6504
9	− 0.8258	+ 0.1542	13	+ 2.9312	+ 1.6659
10	− 2.2782	+ 4.0909	14	+ 13.8108	− 4.7136
11	+ 7.5528	+ 21.2754	15	+ 18.8912	− 45.3175
12	+ 80.1228	+ 29.9663	16	− 33.6045	− 103.0926
13	+ 213.5595	− 79.1750	17	+ 0.5979	− 0.8240
14	+ 2.4126	− 0.7986	18	− 2.3821	+ 1.7223
15	− 3.3950	− 4.3820	19	− 0.0594	+ 0.0637
16	− 0.1882	− 0.1086	20	+ 0.0383	+ 0.0306
17	− 0.0542	+ 0.0745	21	+ 0.0020	− 0.0008
18	+ 0.0022	+ 0.0057	22	+ 0.0002	− 0.0006
19	+ 0.0012	+ 0.0003	23	− 0.0005	+ 0.0003
20	− 0.0008	+ 0.0002	−17+10	+ 0.0008	− 0.0004
−14+ 7	+ 0.0010	− 0.0057	11	− 0.0140	+ 0.0091
8	− 0.0106	− 0.0147	12	− 0.0168	+ 0.1066
9	− 0.1292	− 0.0195	13	+ 0.3910	+ 0.4555
10	− 0.5932	+ 0.5106	14	+ 2.9703	− 0.0100
11	+ 0.0550	+ 4.2398	15	+ 8.2208	− 9.3011
12	+ 14.9414	+ 12.8237	16	− 4.6588	− 39.9350
13	+ 70.8306	− 10.9530	17	− 61.7008	− 56.9620
14	+ 115.8324	− 136.0136	18	+ 0.3273	− 0.8206
15	+ 1.4579	− 0.9192	19	− 0.9874	+ 2.1408
16	− 4.1575	− 1.7500	20	− 0.0205	+ 0.0601
17	− 0.1608	− 0.0103	21	+ 0.0386	+ 0.0072

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse fifth power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
—17+22	+ 0.0018	— 0.0002	—20+24	+ 0.0036	— 0.0215
23	0.0000	- 0.0003	25	— 0.0006	— 0.0008
24	— 0.0004	+ 0.0003	26	— 0.0004	+ 0.0001
—18+11	— 0.0013	— 0.0008	27	— 0.0001	0.0000
12	— 0.0071	+ 0.0148	—21+14	+ 0.0003	+ 0.0010
13	+ 0.0355	+ 0.0927	15	+ 0.0114	+ 0.0058
14	+ 0.5083	+ 0.1795	16	+ 0.0692	— 0.0258
15	+ 2.2469	— 1 2944	17	+ 0.1268	— 0.3562
16	+ 2.1494	- 10.2641	18	— 0.8055	— 1.4460
17	— 19.2032	- 26.4948	19	— 5.9333	— 1.3774
18	— 63.0207	— 15.1096	20	— 14.2028	+ 9.4897
19	+ 0.0514	— 0.7883	21	— 7.9673	+ 28.1476
20	+ 0.1501	+ 1.8777	22	— 0.4890	— 0.2235
21	+ 0.0042	+ 0.0471	23	+ 0.9396	— 0.0418
22	+ 0.0309	— 0.0115	24	+ 0.0173	— 0.0021
23	+ 0.0013	— 0.0004	25	— 0.0055	+ 0.0810
24	— 0.0001	0.0000	26	+ 0.0001	+ 0.0166
25	— 0.0006	+ 0.0002	27	— 0.0002	+ 0.0015
—19+12	— 0.0010	+ 0.0002	28	+ 0.0004	0.0000
13	+ 0.0010	+ 0.0150	—22+15	+ 0.0002	+ 0.0008
14	+ 0.0703	+ 0.0576	16	+ 0.0139	— 0.0006
15	+ 0.4734	— 0.0872	17	+ 0.0076	— 0.0520
16	+ 1.1342	— 1.9401	18	— 0.0590	— 0.3217
17	— 2.6885	— 8.3803	19	— 1.2038	— 0.7290
18	— 24.0400	— 11.1036	20	— 4.8200	+ 1.4254
19	— 47.9844	+ 13.5290	21	— 6.1820	+ 12.1431
20	— 0.2098	— 0.6725	22	+ 5.1793	+ 21.7115
21	+ 0.8373	+ 1.2426	23	— 0.4595	+ 0.0088
22	+ 0.0158	+ 0.0302	24	+ 0.6297	— 0.3905
23	+ 0.0171	— 0.0205	25	+ 0.0144	+ 0.0002
24	+ 0.0003	— 0.0015	26	— 0.0108	— 0.0090
25	— 0.0004	— 0.0003	27	— 0.0008	— 0.0006
26	0.0000	+ 0.0009	28	— 0.0001	+ 0.0003
—20+13	— 0.0010	+ 0.0005	29	+ 0.0004	0.0000
14	+ 0.0077	+ 0.0114	—23+16	+ 0.0010	+ 0.0005
15	+ 0.0815	+ 0.0129	17	+ 0.0087	— 0.0071
16	+ 0.3246	— 0.2755	18	+ 0.0127	— 0.0385
17	+ 0.0198	— 1.9348	19	— 0.1815	— 0.2166
18	— 5.4015	- 4.9725	20	— 1.1884	— 0.0322
19	— 21.2524	+ 1.6657	21	— 2.8686	+ 2.9670
20	— 27.0530	+ 27.0268	22	+ 0.4315	+ 10.8178
21	— 0.4065	— 0.4705	23	+ 11.5092	+ 12.5025
22	+ 1.0620	+ 0.5356	24	— 0.3418	+ 0.1165
23	+ 0.0182	+ 0.0161	25	+ 0.2809	+ 0.2164

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse fifth power, etc.—Continued.

i' i_4	24a		24b	
$-23+26$	+	0.0107	+	0.0588
27	−	0.0112	+	0.0008
28	−	0.0001	−	0.0002
29	−	0.0002	−	0.0001
30		0.0000		0.0000
$-24+17$	+	0.0002	−	0.0004
18	+	0.0044	−	0.0086
19	−	0.0182	−	0.0485
20	−	0.2287	−	0.0856
21	−	0.8847	+	0.4722
22	−	0.8460	+	3.2665
23	+	4.4825	+	7.3042
24	+	12.2651	+	4.0198
25	−	0.1860	+	0.2620
26	−	0.0037	−	0.4572
27	+	0.0066	−	0.0055
28	−	0.0083	+	0.0028
29	−	0.0009	+	0.0001
30	+	0.0001	+	0.0002
31		0.0000		0.0000
$-25+18$	−	0.0008	−	0.0001
19	−	0.0004	−	0.0090
20	−	0.0361	−	0.0290
21	−	0.2060	+	0.0349
22	−	0.4475	+	0.7081
23	+	0.7090	+	2.6536
24	+	5.9176	+	3.2976
25	+	9.2026	−	1.8788
26	+	4.4205	+	0.2617
27	+	0.2220	−	0.3098
28	+	0.0222	−	0.0068
29	−	0.0050	+	0.0052
30	−	0.0004	+	0.0005
31	+	0.0003	+	0.0001
32		0.0000		0.0000
$-26+19$	+	0.0004	−	0.0010
20	−	0.0041	−	0.0070
21	−	0.0395	−	0.0086
22	−	0.1390	+	0.1130
23	−	0.0342	+	0.6967
24	+	1.5590	+	1.5903
25	+	5.3153	−	0.0059
26	+	5.6498	−	4.7876
27	+	0.0365	+	0.2030
28	+	0.1322	−	0.1417
$-26+29$	+	0.0367	−	0.0058
30	+	0.0012	+	0.0054
31	−	0.0003	+	0.0002
32	−	0.0001	−	0.0001
$-27+20$	+	0.0006	−	0.0009
21	−	0.0077	−	0.0030
22	−	0.0329	+	0.0129
23	−	0.0550	+	0.1397
24	+	0.2448	+	0.5785
25	+	2.1273	−	0.2278
26	−	0.8251	−	2.1119
27	+	1.5586	−	4.9265
28	+	0.1114	−	3.1374
29	−	0.2166	−	0.3201
30	−	0.0015	−	0.0207
31	+	0.0012	+	0.0045
32	−	0.0002	−	0.0002
$-28+21$	+	0.0004	+	0.0010
22	−	0.0060	+	0.0002
23	−	0.0188	+	0.0226
24	+	0.0193	+	0.1278
25	+	0.3984	+	0.2640
26	+	1.4068	−	0.3300
27	+	1.6985	−	2.7895
28	−	0.6368	−	3.9551
29	+	0.1357	−	2.1250
30	−	0.1490	−	0.1375
31	−	0.0025	−	0.0146
32	+	0.0023	+	0.0019
$-29+22$	−	0.0006	−	0.0003
23	−	0.0046	+	0.0024
24	−	0.0044	+	0.0348
25	+	0.0670	−	0.0128
26	+	0.3912	+	0.0340
27	+	0.8526	−	1.1025
28	+	0.0972	+	0.7070
29	−	1.9505	−	2.0906
30	+	0.1078	−	0.9708
31	−	0.0712	+	0.0102
32	−	0.0025	−	0.0068
$-30+23$	−	0.0004	−	0.0007
24	−	0.0025	+	0.0042
25	+	0.0078	+	0.0215
26	+	0.0825	+	0.0348

VENUS AND THE EARTH—Concluded.

Numerical development of 24 times the inverse fifth power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
$-30+27$	$+$ 0.3264	$-$ 0.1291	$-31+30$	$-$ 1.2740	$+$ 0.1109
28	$-$ 0.0875	$-$ 1.1000	31	$-$ 1.7805	$+$ 0.2964
29	$-$ 0.9342	$+$ 0.4018	32	$+$ 0.0259	$-$ 0.0514
30	$-$ 2.2000	$-$ 0.7114	$-32+25$	$+$ 0.0002	$+$ 0.0007
31	$+$ 0.0654	$-$ 0.0480	26	$+$ 0.0018	$+$ 0.0029
32	$-$ 0.0066	$+$ 0.0995	27	$+$ 0.0155	$+$ 0.0038
$-31+24$	$-$ 0.0003	$+$ 0.0005	28	$+$ 0.0505	$-$ 0.0388
25	$+$ 0.0001	$+$ 0.0052	29	$+$ 0.0205	$-$ 0.2132
26	$+$ 0.0142	$+$ 0.0034	30	$-$ 0.3852	$-$ 0.4444
27	$+$ 0.0758	$-$ 0.0108	31	$-$ 1.1826	$-$ 0.0906
28	$+$ 0.1510	$-$ 0.2112	32	$-$ 1.0808	$+$ 0.7648
29	$-$ 0.1400	$-$ 0.8247			

MARS AND THE EARTH.

In the following developments of the inverse third power of the distance of Mars from the Earth, I have given the development for ρ_0 as well as for ρ itself, in order that a comparison of the two sets of numbers might afford a general control upon their accuracy, and facilitate a judgment as to the magnitude of the difference between them. It will be seen that, notwithstanding the considerable eccentricity of Mars, the two developments are appreciably different only for the smaller indices, and in nearly all cases so small as to be almost without influence in the planetary theories.

As already remarked, the change from eccentric to mean anomaly was completed only for a portion of the work. So far as completed the corresponding development is given in the margin of the table.

MARS AND THE EARTH.

Development of 24 times the inverse third power of the distance of Mars from the Earth in terms of the eccentric anomaly of the Earth and the mean anomaly of Mars, and also in terms of the mean anomalies of both planets in their elliptical orbits for 1800.

$$\left(\frac{1}{\rho_0}\right)^3=\sum_{i=0}^{i=\infty}\sum_{j=-\infty}^{j=+\infty}\left\{a''_{i,j}\cos\left(i\,u'+(j-i)\,g_4\right)+b''_{i,j}\sin\left(i\,u'+(j-i)\,g_4\right)\right\}$$

$$\left(\frac{1}{\rho}\right)^3=\sum_{i=0}^{i=\infty}\sum_{=-\infty}^{j=+\infty}\left\{a'_{i,j}\cos\left(i\,u'+(j-i)\,g_4\right)+b'_{i,j}\sin\left(i\,u'+(j-i)\,g_4\right)\right\}$$

$$\left(\frac{1}{\rho}\right)^3=\sum_{i=0}^{i=\infty}\sum_{i'=-\infty}^{i'=\infty}\left\{a_{i,i'}\cos\left(i\,g_4+i'\,g'\right)+b_{i,i'}\cos\left(i\,g_4+i'\,g'\right)\right\}$$

$i,\ j$	$24a''$	$24b''$	$24a'$	$24b'$
0 0	+25.9965	. . . . . .	+25.9967	. . . . .
1	+16.0792	− 1.3987	+ 16.0799	− 1.3985
2	+ 4.3145	− 0.7339	4.3173	− 0.7325
3	+ 1.0612	− 0.2657	1.0626	− 0.2653
4	+ 0.2457	− 0.0806	0.2161	− 0.0805
5	+ 0.0544	− 0.0215	+ 0.0544	− 0.0215
1 −5	− 0.0278	− 0.0960	− 0.0279	− 0.0961
−4	− 0.1420	− 0.3932	− 0.1422	− 0.3935
−3	− 0.6703	− 1.5065	− 0.6710	− 1.5070
−2	− 2.8373	− 5.2622	2.8387	− 5.2626
−1	−10.2120	−15.8520	−10.2124	−15.8519
0	−26.5592	−34.8581	−26.5595	−34.8582
1	− 7.5230	− 8.0352	− 7.5237	− 8.0357
2	− 2.0451	− 1.8484	− 2.0473	− 1.8511
3	− 0.5149	− 0.3985	− 0.5160	− 0.3996
4	− 0.1220	− 0.0820	− 0.1223	− 0.0822
5	− 0.0272	− .0164	− 0.0272	− 0.0164
2 −5	− .0918	+ 0.1264	− 0.0919	+ 0.1265
−4	− 0.3252	+ 0.5150	− 0.3254	+ 0.5154
−3	− 1.0379	+ 1.9367	− 1.0384	+ 1.9376
−2	− 2.8729	+ 6.4649	− 2.8734	+ 6.4661
−1	− 6.3452	+17.7462	6.3453	+17.7462
0	− 8.9240	+32.5040	− 8.9242	+32.5043
1	− 0.9780	+ 6.7278	− 0.9785	+ 6.7289
2	− 0.1043	+ 1.6219	− 0.1051	+ 1.6248
3	+ 0.0037	+ 0.3675	+ 0.0037	+ 0.3684
4	+ .0062	+ 0.0794	+ 0.0062	+ 0.0796
5	+ .0022	+ 0.0160	+ 0.0022	+ 0.0160
3 −5	+ 0.2253	− 0.0056	+ 0.2255	− 0.0057
−4	+ 0.8029	− 0.0758	+ 0.8033	− 0.0761
−3	+ 2.6005	− 0.4338	+ 2.6015	− 0.4344

$i,\ i'$	$24a$	$24b$
0 0	+26.0599	. . . . .
1	− 7.5211	− 8.0626
2	− 0.1722	+ 1.5602
3	+ 0.1525	− 0.0987
4	− 0.0161	− 0.0061
−1 −2	− 0.0135	− 0.3535
−1	− 2.0472	+ 1.8572
0	+16.3201	+ 1.6763
1	−26.5411	−34.9688
2	− 1.2202	+ 6.4469
3	+ 0.7365	− 0.4131
4	− 0.0832	− 0.0358
−2 −1	− 0.5160	+ 0.3996
0	+ 4.4030	+ 0.8656
1	−10.0615	−16.3968
2	− 9.0903	+32.4121
3	+ 3.1512	− 1.4397
4	− 0.4008	− 0.1888
5	+ 0.0075	+ 0.0120
−3 0	+ 1.0850	+ 0.3095
1	− 2.7298	− 5.5612
2	− 6.9476	+17.9297
3	+ 22.7839	− 8.9215
4	− 1.2838	− 0.8245
5	+ 0.0463	+ 0.2290
−4 1	− 0.6211	− 1.6149
2	− 3.2924	+ 6.5805
3	+16.7239	− 4.8102
4	−14.5642	− 8.9784
5	− 0.1007	+ 0.4246

MARS AND THE EARTH—Continued.

Development of 24 times the inverse third power of the distance, etc.—Continued.

i, j	24*a''*	24*b''*	24*a'*	24*b'*	*i, i'*	24*a*	24*b*
3−2	+7.3054	−1.7673	+7.3065	−1.7680			
−1	+16.2795	−5.2186	+16.2797	−5.2187	−5 2	−1.2267	+1.9740
0	+22.8424	−9.2464	+22.8428	−9.2466	3	+7.6854	−1.4500
1	+3.2869-	−1.9896	+3.2882	−1.9900	4	−11.7801	−8.8138
2	+0.7506	−0.5225	+0.7528	−0.5235	5	+1.0515	+11.5392
3	+0.1541	−0.1248	+0.1545	−0.1253	6	−0.0018	+0.2386
4	+0.0285	−0.0276	+0.0285	−0.0277			
5	+0.0039	−0.0054	+0.0039	−0.0054	−6 2	−0.3928	+0.5241
					3	+2.7871	−0.2638
4−5	−0.1847	−0.2282	−0.1848	−0.2282	4	−5.6791	−5.1302
−4	−0.6650	−0.7136	−0.6654	−0.7136	5	+0.0558	+11.8322
−3	−2.1220	−1.9780	−2.1226	−1.9780	6	+5.6758	−5.0841
−2	−5.7326	−4.6278	−5.7333	−4.6279	. . . .	. . . . . .	. . . . . .
−1	−11.9165	−8.2846	−11.9166	−8.2848	. . . .	. . . . . .	. . . . . .
0	−14.9733	−8.8283	−14.9737	−8.8285	. . . .	. . . . . .	. . . . . .
1	−1.8541	−0.5862	−1.8551	−0.5867	. . . .	. . . . . .	. . . . . .
2	−0.4792	−0.1452	−0.4807	−0.1461	. . . .	. . . . . .	. . . . . .
3	−0.1016	−0.0240	−0.1019	−0.0241	. . . .	. . . . . .	. . . . . .
4	−0.0200	−0.0034	−0.0200	−0.0034	. . . .	. . . . . .	. . . . . .
5	−0.0043	−0.0010	−0.0043	−0.0010	. . . .	. . . . . .	. . . . . .
5−5	−0.0831	+0.3494	−0.0831	+0.3493	. . . .	. . . . . .	. . . . . .
−4	−0.1811	+1.0855	−0.1811	+1.0852	. . . .	. . . . . .	. . . . . .
−3	−0.2861	+2.9698	−0.2861	+2.9694	. . . .	. . . . . .	. . . . . .
−2	−0.1758	+6.7908	−0.1757	+6.7905	. . . .	. . . . . .	. . . . . .
−1	+0.5333	+11.7270	+0.5334	+11.7272	. . . .	. . . . . .	. . . . . .
0	+1.4477	+11.8261	+1.4476	+11.8265	. . . .	. . . . . .	. . . . . .
1	+0.3981	+0.7178	+0.3981	+0.7186	. . . .	. . . . . .	. . . . . .
2	+0.0966	+0.2528	+0.0967	+0.2540	. . . .	. . . . . .	. . . . . .
3	+0.0219	+0.0478	+0.0220	+0.0479	. . . .	. . . . . .	. . . . . .
4	+0.0043	+0.0091	+0.0043	+0.0091	. . . .	. . . . . .	. . . . . .
5	+0.0005	+0.0018	+0.0005	+0.0018	. . . .	. . . . . .	. . . . . .
6−5	+0.3778	−0.1693	+0.3778	−0.1693	. . . .	. . . . . .	. . . .
−4	+1.0323	−0.5497	+1.0323	−0.5497	. . . .	. . . . . .	. . . . . .
−3	+2.4634	−1.5352	+2.4634	−1.5352	. . . .	. . . . . .	. . . . . .
−2	+4.8336	−3.5029	+4.8336	−3.5029	. . . .	. . . . . .	. . . . . .
−1	+7.0007	−5.8793	+7.0009	−5.8794	. . . .	. . . . . .	. . . . . .
0	+5.6749	−5.5859	+5.6753	−5.5862	. . . .	. . . . . .	. . . . . .
1	−0.0531	−0.2505	−0.0525	−0.2510	. . . .	. . . . . .	. . . . . .
2	+0.0812	−0.1263	+0.0818	−0.1269	. . . .	. . . . . .	. . . . . .
3	+0.0112	−0.0220	+0.0112	−0.0220	. . . .	. . . . . .	. . . . . .
4	+0.0029	−0.0043	+0.0029	−0.0043	. . . .	. . . . . .	. . . . . .
5	−0.0002	−0.0005	−0.0002	−0.0005	. . . .	. . . . . .	. . . . . .

MARS AND THE EARTH—Continued.

Development of 24 times the inverse third power of the distance, etc.—Continued.

i, *j*	24*a*″	24*b*″	24*a*′	24*b*′	*i*, *i*′	24*a*	24*b*
7—5	— 0.3956	— 0.2145	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—4	— 1.0763	— 0.4919	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—3	— 2.5265	— 0.9514	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—2	— 4.8311	— 1.4461	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—1	— 6.7382	— 1.4989	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
0	— 5.1504	— 0.7046	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
1	+ 0.1456	+ 0.1729	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
2	— 0.0895	+ 0.0009	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
3	— 0.0113	+ 0.0020	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
4	— 0.0021	+ 0.0003	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
5	— 0.0002	+ 0.0001	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
8—5	+ 0.073032	+ 0.462748	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—4	+ 0.254889	+ 1.118259	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—3	+ 0.696699	+ 2.313555	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—2	+ 1.460949	+ 3.856621	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—1	+ 2.139512	+ 4.604768	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
0	+ 1.655980	+ 2.871084	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
1	— 0.050326	— 0.300910	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
2	+ 0.032403	+ 0.051113	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
3	+ 0.003523	+ 0.002867	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
4	+ 0.000481	+ 0.000689	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
5	— 0.000018	+ 0.000154	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
9—6	+ 0.134032	0.123011	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—5	+ 0.323465	— 0.339440	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—4	+ 0.685544	— 0.824099	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—3	+ 1.225756	— 1.696901	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—2	+ 1.730192	— 2.786337	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—1	+ 1.690033	— 3.234444	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
0	+ 0.776387	— 1.910446	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
1	— 0.184523	+ 0.252098	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
2	+ 0.020274	— 0.042087	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
3	— 0.000384	— 0.001129	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
4	+ 0.000240	0.000323	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
5	+ 0.000109	+ 0.000022	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
10—6	— 0.184043	— 0.032609	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—5	— 0.450811	— 0.049228	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—4	— 0.969895	— 0.040121	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—3	— 1.762504	+ 0.047783	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—2	— 2.533631	+ 0.247898	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
—1	— 2.529947	+ 0.443574	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
0	— 1.200805	+ 0.344079	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
1	+ 0.276643	— 0.021898	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
2	— 0.038781	+ 0.005340	. . . . .	. . . . .	. . . .	. . . . .	. . . . .

MARS AND THE EARTH—Continued.

Development of 24 times the inverse third power of the distance, etc.—Continued.

$i,\ j$	$24a''$	$24b''$	$24a'$	$24b'$	$i,\ i'$	$24a$	$24b$
10+3	+ 0.000972	+ 0.000201	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
4	− 0.000165	− 0.000065	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
5	− 0.000025	− 0.000056	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
11−7	+ 0.02926	+ 0.06708	. . . . .	. . . .	. . . .	. . . . . .	. . . . .
−6	+ 0.08532	+ 0.16503	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
−5	+ 0.22006	+ 0.36400	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
−4	+ 0.49059	+ 0.70020	. . . . .	. . . .	. . . .	. . . . . .	. . . . .
−3	+ 0.91084	+ 1.12669	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
−2	+ 1.31959	+ 1.41357	. . . .	. . . .	. . . .	. . . . . .	. . . . .
−1	+ 1.30754	+ 1.19726	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
0	+ 0.59625	+ 0.42722	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
1	− 0.15734	− 0.16330	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
2	+ 0.02311	+ 0.02453	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
3	− 0.00116	− 0.00135	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
4	− 0.00001	+ 0.00016	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .
5	− 0.00006	+ 0.00004	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .

THE EARTH AND VENUS.

Coefficients of Cosines.

g' g_4	$(x'+x_4)^2$	$(y'+y_4)^2$	z_4^2	$2(x'+x^4)(y'+y^4)$	$2(y'+y_4)z_4$	$2(x'+x_4)z_4$
0 0	+0.761716	+ .760708	+ .000916	− .000823	− .028096	− .012862
1	+ .025365	− .000771	− 18	+ .008596	+ .000310	− .000930
2	+ .136256	− .136439	+ 279	+ .445509	+ .003702	− .030734
3	+ .000936	− .000938	+ 1	+ .003060	+ 26	− 212
4	+ .000007	− .000006	. . . . . .	+ 20	0	0
1−3	− .000011	− .000010	. . . . . .	0	0	0
−2	− .002166	− .002165	. . . . . .	+ .000003	+ .000087	+ .000119
−1	− .630490	− .630184	. . . . . .	+ .001214	+ .025218	+ .034576
0	− .028982	− .008394	. . . . . .	+ .007269	0	− .000712
1	− .630490	+ .630184	. . . . . .	− .706530	− .025218	+ .034576
2	− .002166	+ .002165	. . . . . .	− .002427	− .000087	+ .000119
3	− .000011	+ .000010	. . . . . .	− .000012	0	0
2−2	− .000018	− .000018	. . . . . .	0	+ 1	+ .000001
−1	− .005293	− .005291	. . . . . .	+ .000011	+ .000212	+ .000290
0	+ .499687	− .499718	. . . . . .	+ .000061	0	− .000006
1	− .005293	+ .005291	. . . . . .	− .005933	− 212	+ .000290
2	− .000018	+ .000018	. . . . . .	− .000020	0	+ .000001
3−1	− .000067	− .000067	. . . . . .	0	+ .000003	+ .000004
−0	+ .008391	− .008391	. . . . . .	+ .000001	0	0
1	− .000067	+ .000067	. . . . . .	− .000074	− .000003	+ .000004
4 0	+ .000141	− .000141	. . . . . .	0	0	0

Coefficients of Sines.

0 1	− .013230	− .004582	− .000017	+ .026132	+ .000634	− .001195
2	− .223220	+ .222286	+ 872	+ .272693	− .030690	− .003668
3	− .001534	+ .001528	6	+ .001874	− .000208	− .000026
4	− .000008	+ .000008	0	+ .000012	0	0
1−3	− .000006	− .000006	. . . . . .	− .000001	+ .000119	0
−2	− .001216	− .001212	. . . . . .	− .000001	+ .034573	− .000087
−1	− .353896	− .352634	. . . . . .	− .000217	+ .000712	− .025220
0	0	+ .007268	. . . . . .	− .037374	+ .034573	0
1	+ .353896	− .352634	. . . . . .	−1.260673	+ .000119	+ .025220
2	+ .001216	− .001212	. . . . . .	− .004331	+ .000119	+ .000087
3	+ .000006	− .000006	. . . . . .	− .000021	0	0
2−2	− .000010	− .000010	. . . . . .	0	0	0
−1	− .002971	− .002961	. . . . . .	− .000002	+ .000290	− .000212
0	0	+ .000061	. . . . . .	+ .999405	− .000006	0
1	+ .002971	− .002961	. . . . . .	− .010584	+ .000290	+ .000212
2	+ .000010	− .000010	. . . . . .	− .000036	0	0
3−1	− .000037	− .000037	. . . . . .	0	+ .000004	− .000003
0	0	0	. . . . . .	+ .016783	0	0
1	+ .000037	− .000037	. . . . . .	− .000134	+ .000004	+ .000003
4 0	0	0	. . . . . .	+ .000282	0	0

The accurate determination with all necessary numerical precision of the coefficients K_1, K_2, etc., of the preceding theory, forms one of the most difficult parts of the present problem. The analytic developments which the author has generally used in the planetary theories seem to be unsatisfactory here, owing to the great number of terms to be included and the slow convergence of the series to be used. He therefore determined to make use of the CAUCHY-HANSEN method, as developed in the preceding section. But in practically applying this method, an unforeseen difficulty occurred. In forming the necessary products of the terms of ρ^5, by $(x+x')^2$, etc., it was found that the final coefficients were much smaller in absolute value than the separate value of the terms whose sum made them up. For example, in the case of HANSEN'S great inequality, the addition of the several numbers which gave the principal coefficients for the argument $18n_4-16n'$ were as follows:

	$\frac{(x_4+x')^2}{\rho^5}$		$\frac{(y_4+y')^2}{\rho^5}$	
	cos	sin	cos	sin
Positive terms .	+63.0215	+46.6846	+62.9561	+46.6238
Negative terms	−63.1753	−46.6273	−63.0257	−46.5312
Difference . . .	− 0.1538	+ 0.0573	− 0.0696	+ 0.0926

We shall hereafter see that the degree of precision thus obtained is not entirely satisfactory. Where each quantity depends on the result of so great a number of operations and additions as is involved in the numerical processes which have to be gone through with, entire accuracy can be secured only by carrying the individual computations to a higher degree of precision than is expected in the final result. It must therefore be regarded as an open question whether the developments in powers of the inclination, as used by DELAUNAY, or the preceding numerical method of HANSEN, is the best. Notwithstanding the difficulties the author inclines to prefer the latter, on the ground that the numerical work may be done by ordinary computers, and believes that, had he been aware of the difficulty in question before the computations were commenced, he could have had them so conducted as to lead to undoubted results.

As the degree of precision obtained is practically sufficient for the present needs of the lunar theory, he has not deemed it necessary to go through so complete a revision of the numerical work as would be necessary to make the expressions reliable beyond the fourth decimal figure. He believes that the coefficients given in the preceding section for inverse powers of the distances of Venus and Mars from the Earth are correct within a very few units in the fourth place of decimals.

It is to be remarked in this connection that the developments of $b_n^{(i)}$ sin i Q, and $b_n^{(i)}$ cos i Q, when the numerical values of these quantities are computed for as many as 24 equidistant intervals admit of a very certain numerical control, in that the sum of the values of any of these functions corresponding to even multiples of 15 degrees, should be nearly equal to the corresponding sum of the odd multiples, the equality probably holding true up to at least the fourth place of decimals. This amounts to the same thing as saying that in the developments the coefficients of the sines and cosines of multiples of g', higher than the 11th, should vanish. From the manner in which these coefficients diminish it would seem that this should be true to at least the fourth place of decimals. As a matter of fact, the actual delopments give coefficients for the cosine of $12g'$, generally amounting to two or three units of the fourth place. This limit, therefore, seems to be that of the precision obtained in the final results.

Of one substantial result there can be no doubt. The developments in question are carried far enough to detect any inequality of long period which can possibly arise, and to assign a limit to the probable order of magnitude of its amplitude. In the present state of the lunar theory, when observations of the Moon seem to show inequalities still unaccounted for, this result may be regarded as a point gained.

§ 19.

INEQUALITIES INDEPENDENT OF R.

In the preceding theory the inequalities of the lunar elements determined by the three equations (94) reduced to numbers in (96)

$$\begin{aligned}
\delta\, \nu &= -.0096\ \delta\nu' + .003\,04\ \delta.\, e'^2 \\
\delta\, e^2 &= -.000\,58\ \delta\nu' - .000\,09\ \delta.\, e'^2 \\
\delta.\gamma^2 &= \qquad\qquad\quad\ - .000\,011\ \delta.\, e'^2
\end{aligned}$$

are not confined to the secular variation. We must, in fact, substitute for $\delta\nu'$ and $\delta.\, e'^2$ the entire inequalities of these elements of the Earth's orbit produced by the action of the planets. As this operation is quite independent of determining the other inequalities, I have brought the results together in a single section. The computation of $\delta\nu'$ and $\delta.e'^2$ is so simple that I do not deem it necessary to enter into details of the computation, and shall therefore only give the results.

Putting λ_4, λ', the mean longitudes of Venus and the Earth, counted from the ascending node of the orbit of Venus, I have found the following results:

From the action of Venus.

$$\begin{aligned}
\delta\nu' = &+1.''52\ \cos(\lambda' - \lambda_4) \\
&-0.''83\ \cos(2\lambda' - 2\lambda_4) \\
&-0.''51\ \cos(3\lambda' - 3\lambda_4) \\
&+0.''06\ \cos(2\lambda' - \lambda_4) - 0.''01\ \sin(2\lambda' - \lambda_4) \\
&-0.''29\ \cos(3\lambda' - 2\lambda_4) - 0.''07\ \sin(3\lambda' - 2\lambda_4) \\
&+0.''09\ \cos(5\lambda' - 3\lambda_4) - 0.''01\ \sin(5\lambda' - 3\lambda_4) \\
&-0.''0045\ \cos(13\lambda' - 8\lambda_4) - 0.''0029\ \sin(13\lambda' - 8\lambda_4)
\end{aligned} \qquad (146)$$

$$\begin{aligned}
\delta e' = & +0.''88 \cos \lambda_4 + 0''.41 \sin \lambda_4 \\
& +0.''80 \cos \lambda' + 0''.37 \sin \lambda' \\
& +1.''52 \cos (2\lambda' - \lambda_4) + 0''.71 \sin (2\lambda' - \lambda_4) \\
& -3.''66 \cos (3\lambda' - 2\lambda_4) - 1''.71 \sin (3\lambda' - 2\lambda_4) \\
& +0.''48 \cos (5\lambda' - 3\lambda_4) + 0''.37 \sin (5\lambda' - 3\lambda_4) \\
& -0.''010 \cos (13\lambda' - 8\lambda_4) - 0''.019 \sin (13\lambda' - 8\lambda_4)
\end{aligned}$$

From the action of Mars.

$$\begin{aligned}
\delta v' = & -0.''04 \cos (\lambda_4 - \lambda') \\
& -0.''08 \cos (2\lambda_4 - 2\lambda') \\
& +0.''032 \cos (2\lambda_4 - \lambda' - \varpi') \\
& -0.''069 \cos (2\lambda_4 - \lambda' - \varpi_4) \\
\delta e' = & -0.''95 \cos (2\lambda_4 - \lambda' - \varpi')
\end{aligned}$$

From the action of Jupiter.

$$\begin{aligned}
\delta v' = & +0.''24 \cos (\lambda_4 - \lambda') \\
& -2.''33 \cos (2\lambda_4 - 2\lambda') \\
& +0.''42 \cos (3\lambda_4 - 2\lambda' - \varpi_4) \\
\delta e' = & -0.73 \cos (\lambda' - \varpi') \\
& -3.09 \cos (\lambda_4 - \varpi_4) \\
& +3.88 \cos (2\lambda_4 - \lambda' - \varpi')
\end{aligned}$$

The preceding formulæ then give

Action of Venus on the Moon.

$$\begin{aligned}
\delta \varepsilon = & +0.''46 \sin (\lambda' - \lambda_4) \\
& -0.''13 \sin (2\lambda' - 2\lambda_4) \\
& -0.''05 \sin (3\lambda' - 3\lambda_4) \\
& +0.''22 \sin (3\lambda' - 2\lambda_4) \quad -0.''05 \cos (3\lambda' - 2\lambda_4) \\
& +0.''14 \sin (5\lambda' - 3\lambda_4) \quad +0.''02 \cos (5\lambda' - 3\lambda_4) \\
& +0.''03 \sin (2\lambda' - \lambda_4)
\end{aligned} \tag{147}$$

$$\begin{aligned}
&+0.''20 \sin(13\lambda'-8\lambda_4) && -0.''12 \cos(13\lambda'-8\lambda_4)\\
\delta\pi=&-1.''12 \sin(\lambda'-\lambda_4) &&\\
&+0.''32 \sin(2\lambda'-2\lambda_4) &&\\
&-0.''07 \sin(2\lambda'-\lambda_4) &&\\
&+0.''12 \sin(3\lambda'-3\lambda_4) &&\\
&-0.''54 \sin(3\lambda'-2\lambda_4) && +0.''12 \cos(3\lambda'-2\lambda_4)\\
&-0.''35 \sin(5\lambda'-3\lambda_4) && -0.''05 \cos(5\lambda'-3\lambda_4)\\
&-0.''50 \sin(13\lambda'-8\lambda_4) && +0.''30 \cos(13\lambda'-8\lambda_4)\\
\delta\theta=&+0.''37 \sin(\lambda'-\lambda_4) &&
\end{aligned}$$

Action of Mars on the Moon.

$$\begin{aligned}
\delta\varepsilon=&+0.''02 \sin(\lambda_4-\lambda')\\
&+0.''13 \sin(2\lambda_4-\lambda'-\omega')\\
&-0.''21 \sin(2\lambda_4-\lambda'-\omega_4)\\
\delta\pi=&-0.''04 \sin(\lambda_4-\lambda')\\
&-0.''31 \sin(2\lambda_4-\lambda'-\omega')\\
&+0.''51 \sin(2\lambda_4-\lambda'-\omega_4)
\end{aligned} \tag{148}$$

Action of Jupiter on the Moon.

$$\begin{aligned}
\delta\varepsilon=&-0.''05 \sin(\lambda_4-\lambda')\\
&+0.''25 \sin(2\lambda_4-2\lambda')\\
&+0.''08 \sin(\lambda_4-\omega')\\
&-0.''05 \sin(3\lambda_4-\lambda'-\omega_4)\\
\delta\pi=&-0.''60 \sin(2\lambda_4-2\lambda')
\end{aligned} \tag{149}$$

If, in these expressions we consider that part of $\delta\varepsilon$ which arises from δv, we shall see that it is necessarily less than the corresponding term in the mean longitude of the Sun. For instance, the term

$$a_c \cos \mathrm{N}$$

in $\delta v'$ leads in $\delta\varepsilon'$ or in $\int n'dt$ to the term

$$-\frac{3}{2}\frac{n'}{b}\,a_e \sin \mathrm{N},$$

while the term produced in $\delta\varepsilon$ is

$$.0144\,\frac{n}{b}\,a_e \sin \mathrm{N},$$

which is about one-eighth that produced in $\delta\varepsilon'$. Hence, if, in our theory, we have omitted sensible terms from this source, terms, eight times as great must be omitted in the theory of the Sun. In this proposition it is assumed that the perturbations of the Earth's eccentricity are so minute that in the expression

$$.000\,155\ \nu\ e_e$$

no omitted term can possibly become sensible.

§ 20.

PRELIMINARY DATA FOR INEQUALITIES ARISING THROUGH R_2.

The following values of the masses of the planets were adopted in the computations:

Mercury,	$\frac{1}{5\,000\,000}$
Venus,	$\frac{1}{420\,000}$
Earth,	$\frac{1}{326\,800}$
Earth + Moon,	$\frac{1}{322\,800}$
Mars,	$\frac{1}{3\,000\,000}$
Jupiter,	$\frac{1}{1047.8}$

The expressions for the action of the planets on the Moon all contain the factor

$$\mu^4\,\alpha^3 = \frac{m_4}{m_2 + m_3}\,\frac{a^3}{a'^3}$$

where a is the mean distance of the Moon from the Earth, and a' the mean distance of the Earth from the Sun. Of this expression the factor

$$\frac{1}{m_2 + m_3}\,\frac{a^3}{a'^3}$$

admits of being found with greater precision than we can find its component parts, because the quantity $\frac{m_2 a'_3}{a^3}$ is derivable from the length of the seconds pendulum, combined with the dimensions of the Earth. These data give the equations

$$\log m_2 a'^3 = 7\ 58838$$

$$\log a = 1.78044$$

taking for $m_3 : m_2$ the value $1:81.4$ we then have

$$\log \frac{m_1}{m_2+m_3} \frac{a_3}{a'^3} = 7{\cdot}74763 - 10$$

a purely numerical quantity, independent of the adopted units of mass and length.

Since we represent by m_4 the ratio of the mass of the planet to that of the Sun we have for the value of the common factor sought

$$\mu_4 \frac{a^3}{a'^3} = [7{\cdot}74763 - 10]\, m_4$$

Referring to the equations (73) and (74) it will be seen that the magnitude of the coefficient of any inequality depends very largely on the value of ν, the ratio of the period of an argument to that of revolution of the Moon. The magnitude of the coefficients $k_1, k_2 \;.\;.\;.\; k_6$ which enter into K is also to be considered. To facilitate the search for inequalities which may become sensible in consequence of a large value of ν the following table of motions of the lunar arguments N, and of the principal term in K_1 and K_5, has been prepared:

Daily motions of the lunar arguments for all terms to those of the third order inclusive, with order of magnitude of the corresponding coefficients.

Argument.	Daily motion.	Principal term of x_1	log x_1
l, π, θ	$''$		
0, 0, 2	−381	$+\gamma^2$	−2.7
0, 2, 0	802	$+\frac{5}{4}e^2$	−2.4
1, 0, 0	47435	$+\frac{45}{32}\alpha m$	−3.6
1, −3, 0	46232	$-\frac{7}{48}e^3$	−4.6
1, −1, 0	47034	$-e$	−1.3
1, −1, −2	47416	$-e\gamma^2$	−4.0
1, −1, 2	46652	$+\frac{11}{4}e\gamma^2$	−3.6
1, 1, 0	47836	$-\frac{3}{2}e$	−1.1
1, 1, −2	48217	$-\frac{1}{2}e\gamma^2$	−4.3
2, 0, 2	94488	$\frac{3}{4}\gamma^2 m$	−3.9
2, 0, 0	94870	$+\frac{1}{2}$	−0.3
2, 0, −2	95251	$+\gamma^2$	−2.7
2, −2, 0	94068	$-\frac{1}{4}e^2$	−3.1
2, 2, 0	95672	$-\frac{15}{16}e^2 m$	−3.7
3, 0, 0	142305	$-\frac{5}{4}\alpha e'$	−4.3
3, −1, 0	141904	$-\frac{1}{2}e$	−1.6
3, −1, −2	142285	$+e\gamma^2$	−4.0
3, 1, 0	142706	$+\frac{15}{16}em$.	−2.4

Daily motions of the lunar arguments, etc.—Continued.

Argument.	Daily motion.	Principal term of $\varkappa_1$	log $\varkappa_1$
l, π, θ	$''$		
3, 1, −2	143087	$-\frac{5}{4}e\gamma^2$	−3.9
3, −3, 0	141102	$-\frac{1}{8}e^3$	−4.7
4, 0, 0	189740	$+\frac{3}{16}m^2$	−3.0
4, −2, 0	188938	$+\frac{3}{8}e^2$	−4.2

Argument.	Daily motion	Principal term of 2 $\varkappa_5$	log 2 $\varkappa_5$
0, 0, 1	−191	-2γ	−1.1
0, 2, −1	993	$-\frac{15}{4}e\gamma^2$	−3.3
1, −1, −1	47225	$-2e\gamma$	−2.3
1, −1, 1	46843	$+2e\gamma$	−2.3
1, 1, −1	48027	$+6e\gamma$	−1.8
1, 1, 1	47645	$-\frac{3}{2}e\gamma m$	−3.6
2, −2, −1	94259	$-\frac{1}{2}e^2\gamma$	−4.2
2, −2, 1	93877	$+\frac{7}{4}e^2\gamma$	−3.6
2, 0, −3	95442	$+2\gamma^3$	−3.7
2, 0, 1	94680	$-\frac{3}{4}\gamma m$	−2.6
2, 0, −1	95061	-2γ	−1.1

Daily motions of the lunar arguments, etc.—Continued.

Argment.	Daily motion.	Principal term of 2 $\varkappa_5$	log 2 $\varkappa_5$
$l,\ \pi,\ \theta,$	$''$		
3, −1, −1	142095	$-2\,e\,\gamma$	−2. 3
3, −1, 1	141713	$-\frac{3}{4}e\,\gamma$	−2. 7
3, 1, −1	142896	$-\frac{15}{4}e\,\gamma\,m$	−3. 2
4, −2, −1	189129	$-2\,e^2\,\gamma$	−3. 6
4, 0, −1	189930	$-\frac{3}{4}\gamma\,m^2$	−3. 7

To discover inequalities of long period it is only necessary to compare the motions of this table with the motions of all possible arguments of the form $i_4\,g_4 - i'\,g'$.

§ 21.

ACTION OF MERCURY.

The adopted mass of Mercury, $\frac{1}{5\,000\,000}$, gives from the numbers of § 20

$$\mu_4\,a^3 = 0''.000\ 231$$

If we substitute this coefficient in (110) and retain only the largest terms in each case, we shall have

$$\delta w = +0''.000\ 233\ i\ \ \nu\ \{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U})+\mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}$$
$$e\,\delta e = -0\ .000\ 122\ i'\ \nu\ \{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U})+\mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}$$
$$\gamma\,\delta\gamma = -0\ .000\ 029\ i''\nu\ \{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U})+\mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}$$

To facilitate an estimate of the order of magnitude of the coefficients of any of these inequalities we recall the values (107) of K and K′

$$\pm\mathrm{K} = \varkappa_1\,\mathrm{K}'_1 + \varkappa_2\,\mathrm{K}'_2 + \ .\ .\ .$$
$$\mathrm{K}' = \varkappa_1\,\mathrm{K}_1 + \varkappa_2\,\mathrm{K}_2 + \ .\ .\ .$$

The largest values of K_1, K_2 . . . K'_1, K'_2 . . . are of the order of magnitude unity for some of the arguments of lowest index, and decrease indefinitely with larger values of the indices.

By inspecting the values of $\varkappa_1$, $\varkappa_2$, etc., on p. 217, etc., it will be seen that the value unity is approximated to only in the terms corresponding to the following system of indices

$$i\ =0;\ 2$$
$$i'\ =0;\ 0$$
$$i''=0;\ 0$$

In the case of the first set of indices the values of δv, δe, and $\delta\gamma$ vanish entirely, and in the case of the second the largest terms can only be of the order of magnitude,

$$0''.0005\ \nu\ \mathrm{K}_i$$

and since K_i never much exceeds unity, this term can become sensible only for a very large value of ν. Now referring to (112) and (113) it will be seen that if ν is very large, the terms in ε_0, ε_s, etc., multiplied by ν^2 will, in general, be much larger than those multiplied by ν, and that in nearly every case the perturbations of ε will be larger than those of π or θ. If, then, there is any sensible inequality of long period arising from the action of Mercury it is to be sought in the terms

$$\varepsilon_0=(0''.000\,35\,i-0''.000\,005\,i'+0''.000\,001\,i'')\,\nu^2\,\mathrm{K}$$

$$\varepsilon_s=(0''.000\,35\,i-0\ .000\,005\,i'+0\ .000\,001\,i'')\,\nu^2\,\mathrm{K}'$$

In order that these coefficients may amount to 0''.1 we must have, for $i=1$

$$\text{when } \mathrm{K}=1, \qquad \nu>15$$
$$\text{“} \quad \mathrm{K}=.01, \qquad \nu>150$$
$$\text{“} \quad \mathrm{K}=.0001, \quad \nu>1500$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

The value of ν is inversely proportional to the coefficient of the time in the angle $\mathrm{N}\pm\mathrm{U}$. These angles are of the form

$$\mathrm{N}=i\,\varepsilon+i'\,\pi+i''\,\theta+i'''\,g'$$
$$\mathrm{U}=j'\,g'+j_4 g_4$$

The angle $\mathrm{N}\pm\mathrm{U}$ will therefore comprise five separate variables, the angle of g' being common to both N and U. It follows that in seeking for large values of ν it is only necessary to consider those values of N in which $i'''=0$, and to find, for each of such values, whether there are any integral values of j' and j_4 which will make the coefficient of the time in $\mathrm{N}\pm\mathrm{U}$ very small. On the next page is found a complete list of the combinations of the indices i, i', and i'', which we have found to appear in the development of x^2, xy, etc., to terms of the third order, with the daily motion of the argument corresponding to each combination.

Opposite each set of indices is given the order of magnitude of the largest corresponding value of κ, which may be regarded as nearly a superior limit of any possible corresponding value of K or K′.

We have next to consider the probable order of magnitude of the coefficients K_1, K_2, etc I find, by induction, that these coefficients are of the same order of magnitude with those which we meet in the development of $\frac{1}{\rho_3}$, a result which might have been deduced *a priori*. Comparing the several multiples of the mean motions we find that within these limits those in the following table are all that nearly coincide with any one of the preceding values of $\frac{n}{\nu}$. With each combination of arguments is given a computation of the order of magnitude of the corresponding term in $\delta\varepsilon$, or, rather, a computation showing a value which the coefficient of the term is not likely to exceed. The third column shows the daily motion of the argument corresponding to the indices in the first two columns, or the value of

$$i n + i'\pi_1 + i''\theta_1 \pm (j' n' + j_4 n_4)$$

We then have, in the fourth column, the approximate logarithm of ν^3.

In the fifth is log κ_i, concluded from the order of magnitude of the largest value of κ in the preceding table.

In the sixth is the largest probable value of log κ_i, concluded by induction from the magnitude of the terms in the development of $\frac{1}{\rho^3}$.

In the seventh and eighth are the resulting largest probable value of ε_c or ε_s, concluded from the formulæ (113)

		Daily motion.	log ν^2	log κ_i	log K_i	log ε_c	ε_c
i i' i''	j' j_4	″				+10	″
1 1 0	+ 1 + 3	90	5.5	−1.1	−2.	9.0	0.1
1 1 0	− 3 + 4	449	4.1	−1.1	−1.	8.6	0.04
1 1 −2	− 3 +.4	68	5.7	−4.3	−1.	7.0	0.001
2 −2 0	−15 +10	33	6.4	−3 1	−5.	5.0	0.000
3 −1 0	−14 +13	58	5.9	−1.6	−4.	6.9	0.001
3 0 0	−18 +14	81	5.6	−4.3	−5.	2.9	0.000
1 1 −1	− 3 + 4	258	4.6	−1.8	−1.	8.4	0.02
1 1 1	+ 1 + 3	100	5.4	−3.6	−2.	6.4	0.000

The first three terms are the only ones in which there is any reasonable hope of finding a sensible coefficient, and they may therefore merit a closer examination, although their actual coefficients are probably smaller than those here given. The following are the actual values of the highest terms in K, and the actual value of the coefficient in the development of $\frac{1}{\rho^3}$ for the combinations of arguments which form the highest terms in the final coefficient.

1. For N $= \epsilon+\pi$ For U $=g'+3g_4$

$$\varkappa_1=-\frac{3}{2}e=-.082 \qquad K_i=.0006\pm$$

$$\varkappa_2=\frac{3}{2}e=+.082$$

$$\varkappa_4=-\frac{3}{2}e=-.082$$

$\varkappa_3$, $\varkappa_5$, and $\varkappa_6=0$

2. For N $=\epsilon+\pi-2g'$ For U $=-g'+3g_4$

$$\varkappa_1=\varkappa_2=-\frac{15}{8}em=-.0077 \qquad K_i=.016\pm$$

$\varkappa_3$ to $\varkappa_6=0$

The coefficient corresponding to this value of K_i would be of the order of magnitude 0.″03. It seems to me without the limits of reasonable probability that the actual values of K_1 and K_2 should exceed the value here assigned by more than two or three times, and therefore very unlikely that the coefficient should amount to 0.″1. It is equally probable that the values of the following terms are less than those assigned. I conclude that the motion of the Moon is subject to no sensible inequality of long period arising from the direct action of Mercury.

§ 22.

ACTION OF VENUS.

With the adopted mass 1 ÷ 420 000 we find

$$\mu_4 a^3=0.''002\,746 \quad .$$

By the substitution of this value the secular terms (114) become

$$\delta\epsilon=0.''002\,746\left\{-2.023\ K'_0-1.011\ \frac{dK'_0}{dx}+0.0005\frac{dK'_0}{e\,de}\right\}nt$$

$$\delta\pi=0.''002\,746\left\{+0.0293\,K'_0+0.0147\frac{dK'_0}{dx}+0.526\ \frac{dK'_0}{e\,de}\right\}nt$$

$$\delta\theta=0.''002\,746\left\{-0.0074\,K'_0-0.0037\frac{dK'_0}{dx}+0.125\ \frac{dK'_0}{\gamma\,d\gamma}\right\}nt$$

The largest values of K'_0 and its derivatives will be those which arise from the combination of the zero terms in K_1, K_2, etc., with the values of $\varkappa_1$, $\varkappa_2$, etc., corresponding to $N=0$, and these terms must be doubled, because the angles $N+U$ and $N-U$ will then both be zero. From the developments given on pp. 252–257 and 262, we find the following numerical values of K_1, K_2, etc., or the constant terms in the developments (103). Along side of them we place the corresponding values of $\varkappa_1$, $\varkappa_2$, etc., and their derivatives from the first terms in the analytical tables, pages 217-224.

When $U=0$

$$32\,K_1=+38.98 \qquad \varkappa_1=+0.49947;\ \frac{d\varkappa_1}{dw}=+.0021;$$

$$32\,K_2=+38.06 \qquad \varkappa_2=\varkappa_1 \qquad \frac{d\varkappa_1}{e\,de}=+1.52;\ \frac{d\varkappa_1}{\gamma\,d\gamma}=-1.998$$

$$32\,K_3=-77.10 \qquad \varkappa_3=+0\,00403$$

$$32\,K_4=+57.51 \qquad \varkappa_4=0$$

$$32\,K_5=+19.14 \qquad \varkappa_5=0$$

$$32\,K_6=+\ 8\,76 \qquad \varkappa_6=0$$

An important remark is to be made here. It will be seen that in these and many other terms we have

$$\varkappa_1=\varkappa_2;\quad \varkappa_4=\varkappa_5=\varkappa_6=0$$

The computation of these terms admits of being greatly simplified. If we take the sum of the first three equations (103) noticing that

$$(x'+x_4)^2+(y'+y_4)^2+(z'+z_4)^2=\rho^2$$

we find

$$K_1+K_2+K_3=0$$
$$K'_1+K'_2+K'_3=0$$

two equations which form a valuable check on the accuracy of the computation of K_1, K_2, and K_3. Now, in the case supposed, the value of K and K' in the equations (107) becomes

$$\pm K=(K'_1+K'_2)\,\varkappa_1+K'_3\,\varkappa_3$$
$$K'=(K_1+K_2)\,\varkappa_1+K_3\,\varkappa_3$$

Substituting the value of $K'_1+K'_2$ and of K_1+K_2, given by the preceding equations, these quantities become

$$\left.\begin{aligned} K&=\pm K'_3(\varkappa_3-\varkappa_1)\\ K'&=\ K_3\,(\varkappa_3-\varkappa_1)\end{aligned}\right\}\quad(150)$$

Hence the only values of K_i and K'_i we need in this case are those corresponding to $i=3$, and these are the ones most easily computed, owing to the small number of terms in z_4.

In another extended class of cases we have

$$\varkappa_1=-\varkappa_2=\pm\varkappa_4; \quad \varkappa_2=\varkappa_5=\varkappa_6=0$$

Then (107) gives

$$\begin{aligned} K &=\pm(K'_1-K'_2\pm K_4)\,\varkappa_1 \\ K' &= \quad (K_1-K_2\pm K'_4)\,\varkappa_1 \end{aligned} \tag{151}$$

In (150) and (151) the upper sign of the double ones without the parentheses corresponds to the combination N + U, and the lower to N − U.

The terms now under consideration belong to the former class. Forming the values of K' and its derivatives by (150) and substituting in the preceding expressions we find, doubling the magnitude of the term,

$$\begin{aligned} \tfrac{1}{2}\,\delta\varepsilon &=-0.''00663nt \\ \tfrac{1}{2}\,\delta\pi &=+0.''00539nt \\ \tfrac{1}{2}\,\delta\theta &=-0.''00165nt \end{aligned}$$

Taking the Julian year as the unit of time we shall have $n=84.00$, which will give

$$\begin{aligned} \delta\varepsilon &=-1.''113t \\ \delta\pi &=+0.''905t \\ \delta\theta &=-0.''277t \end{aligned}$$

Now, taking up the periodic terms, the equations (110) become, by substituting the values of $\mu_4 a^3$,

$$\begin{aligned} \delta w &=\{0.''002\,78i-0.00004i'\} && \{\nu K\sin(N\pm U)+\nu K'\cos(N\pm U)\} \\ e\,\delta e &=-0.''001\,444i' && \{\nu K\sin(N\pm U)+\nu K'\cos(N\pm U)\} \\ \gamma\,\delta\gamma &=-0.''000\,343i'' && \{\nu K\sin(N\pm U)+\nu K'\cos(N\pm U)\} \end{aligned}$$

while the coefficients (113) become

$$\begin{aligned} \varepsilon_c=(&\ \ 0.''004\,166i-0.''000\,060i'+0.000\,014i'')\,\nu^2 K \\ &+0.''005\,56\,\nu K+0.''002\,78\,\nu\frac{dK}{dw} \\ \varepsilon_s=(&-0.''004\,166i+0.''000\,060i'-0.''000\,014i'')\,\nu^2 K' \end{aligned}$$

$$-0.''005\,56\,\nu\,\mathrm{K}'-0.''007\,28\,\nu\frac{d\,\mathrm{K}'}{d\,w}$$

$$\pi_c=(-0.''000\,060i-0.''000\,027i'-0.''000\,034i'')\,\nu^2\,\mathrm{K}$$

$$-0.''000\,082\,\nu\,\mathrm{K}-0.''000\,040\,\nu\frac{d\mathrm{K}}{d\,w}-0.''001\,44\,\nu\frac{d\,\mathrm{K}}{e\,d\,e}$$

$$\pi_s=(\;0.''000\,060i+0.''000\,027i'+0.''000\,034i'')\,\nu^2\,\mathrm{K}'$$

$$+0.''000\,082\,\nu\,\mathrm{K}'+0.''000\,040\,\nu\frac{d\,\mathrm{K}'}{d\,w}+0.''001\,44\,\nu\frac{d\,\mathrm{K}'}{e\,d\,e}$$

$$\theta_c=(\;0.''000\,016i-0.''000\,034i'+0.''000\,005i'')\,\nu^2\,\mathrm{K}$$

$$+0.''000\,021\,\nu\,\mathrm{K}+0.''000\,010\,\nu\frac{d\mathrm{K}}{d\,w}-0.''000\,34\,\nu\frac{d\,\mathrm{K}}{\gamma\,d\,\gamma}$$

$$\theta_s=(-0.''000\,016i+0.''000\,034i'-0.''000\,005i'')\,\nu^2\,\mathrm{K}'$$

$$-0.''000\,021\,\nu\,\mathrm{K}-0.''000\,010\,\nu\frac{d\mathrm{K}'}{d\,w}+0.''000\,34\,\nu\frac{d\,\mathrm{K}'}{\gamma\,d\,\gamma}$$

The magnitude of these coefficients will depend mainly upon that of the factors ν and K, or, since the latter is made up of the two factors K_i and $\varkappa_i$, there will be three factors in all, the order of magnitude of which is to be considered. Since the largest value which K can attain is only two or three units, we see at a glance that no term can become sensible unless ν is a large number. Now, there are two classes of terms in which ν may be large, in other words terms, the period of which may be many times that of the Moon, namely:

(1) Those in which i, i', and i'' all vanish, so that the term contains only the mean longitude of the Earth and planet. But in this case the coefficient of $\nu^2\,\mathrm{K}$ will also vanish, so that the perturbation will contain only the first power of ν.

(2) Those in which one or more of the indices i, i', and i'' occur, so that the term contains the square of ν. In all these terms K is necessarily extremely minute, for if the term contains i, the coefficient of the Moon's mean longitude, the coefficient of t in the angle $\mathrm{N}\pm\mathrm{U}$, or

$$i\,l+i'\,\pi+i''\,\theta\pm\mathrm{U},$$

can become very minute only for very high multiples of the mean longitudes of the planet and Earth in U, owing to the magnitude of the mean motion of the Moon compared with that of the Earth and planet. If, on the other hand, i does not occur in the term, the condition of long period may be fulfilled by the lower multiples of U, when K_1, K_2, etc., will be considerable. But in this case two very minute factors will enter in, namely, $\varkappa_1$, $\varkappa_2$, etc., which are very minute when $i=0$, and the coefficient will be only $0.''000060i'$ or $0.''000014i''$.

We begin by giving all the values of K_1, K_2, etc., which have been actually computed, each of them being multiplied by 32. With them are given for comparison, at the bottom of the column, the corresponding coefficient in the development of $\frac{32}{\rho^3}$.

U	$-g'+g_4$		$-2g'+2g_4$		$-3g'+2g_4$	
i	32 K_i	32 K'_i	32 K_i	32 K'_i	32 K_i	32 K'_i
1	+ 60.25	− 34.5	+50.56	− 49.23	+1.75	− 2.09
2	+ 58.92	− 33.3	+ 8.26	− 47.25	+4.19	− 4.56
3	−118.98	+ 66.7	−58.96	+ 96.37	−6.21	+ 6.44
4	− 1.89	+ 0.76	− 1.57	+ 1.32	+0.69	+ 1.53
5	+ 31.61	− 21.62	+14.21	− 34.16	+2.08	− 2.19
6	+ 18.96	− 2.02	+19.39	+ 7.80	+1.27	− 0.53
	+183.06	−102.61	+90.99	−148.73	+9.66	−10.00

In all these lower terms, that part of K_3 which results from the development of $\frac{3}{2}\frac{z_4^2}{\rho^5}$ is much smaller than that which results from the development of $\frac{1}{2}\frac{1}{\rho^3}$ so that we might have rejected the former without sensible error. Rejecting them in the term of which the argument is $-5g'+3g_4$ we shall have for this term

$$32\ K_3 = -0.95$$

$$32\ K'_3\quad +1.03$$

In the term $-3g'+3g$ we shall have

$$32\ K_3 = -3.57$$

$$32\ K'_3 = +93.39$$

On the same supposition we should have for the terms $-13g'+8g_4$,

$$32\ K_3 = +.0021$$

$$32\ K'_3 = +.0016$$

and for $U = -11g'+8g_4$,

$$32\ K_3 = +0.127$$

$$32\ K'_3 = -0.003$$

but owing to the minuteness of these we can not be sure that they express more than the order of magnitude of K_3 and K'_3.

These terms comprise all those independent of i, i', and i'', to which correspond very large values of either K or ν. They are to be combined with the first four terms in the development of x^2, y^2, etc. The values of $\varkappa_1$, $\varkappa'_2$, etc., corresponding to the zero term of this development have been already given. The next largest term is the third, corresponding to $N = 2g'$, for which

$$\varkappa_1 = -\frac{19}{16}m^2 - \frac{97}{24}m^3 + \frac{75}{16}e^2m + \frac{3}{4}\gamma^2m$$

and in numbers

$$\varkappa_1 = -\varkappa_2 = \varkappa_4 = -.007\ 17$$

$$\frac{d\varkappa_1}{dw} = -\frac{d\varkappa_2}{dw} = \frac{d\varkappa_4}{dw} = +.0258$$

$$\frac{d\varkappa_1}{de} = -\frac{d\varkappa_2}{de} = \frac{d\varkappa_4}{de} = +.039$$

$$\frac{d\varkappa_1}{d\gamma} = -\frac{d\varkappa_2}{d\gamma} = \frac{d\varkappa_4}{d\gamma} = +.0050$$

This term can give rise to a sensible result only by being combined with a large term in which the coefficient of the time is nearly equal to $2n'$. The two most favorable ones are

$$U = -3g' + 3g_4 \text{ which gives } N - U = 5g' - 3g_4$$

$$U = -11g' + 8g_4 \quad . \quad . \quad . \quad . \quad N - U = 13g' - 8g_4$$

both of which are of long period.

Our formulæ now gives the following computation of all the terms of the class under consideration which can become sensible.

U	$-g'+g_4$	$-2g'+2g_4$	$-3g'+3g_4$	$-3g'+2g_4$	$-5g'+3g_4$	$-13g'+8g_4$
N	0	0	0	0	0	0
K_3	−3.72	−1.84	−0.11	−0.19	−0.030	−.00065
K'_3	+2.08	+3.01	+2.92	+0.20	+0.032	+.00050
K	∓1.030	∓1.491	∓1.447	∓0.099	∓0.0159	∓.00025
K′	+1.843	+0.912	+0.054	+0.094	+0.0149	−.00032
ν	±21.37	±10.68	±7.12	±53.25	∓108.3	±3196.
ε_c	−0.″1226	−0.″0887	−0.″0574	−0.″0293	+0.″0096	−0.″0045
ε_s	∓0.2190	∓0.0542	∓0.0021	∓0.0278	±0.0090	∓0.0056
π_c	+0.097	+0.071	+0.05			
π_s	±0.175	±0.043	0 00			

Where there are double signs the upper corresponds to the angle $+U$ and the lower to $-U$, and both are to be combined into a single term. Combining them, the results are

$$
\begin{aligned}
\delta l = & -0.''438 \sin(-g'+g_4) - 0.''245 \cos(-g'+g_4) \\
& -0\ .108 \sin(-2g'+2g_4) - 0\ .177 \cos(-2g'+2g_4) \\
& -0\ .004 \sin(-3g'+3g_4) - 0\ .115 \cos(-3g'+3g_4) \\
& -0\ .056 \sin(-3g'+2g_4) - 0\ .059 \cos(-3g'+2g_4) \\
& +0\ .018 \sin(-5g'+3g_4) + 0\ .019 \cos(-5g'+3g_4) \\
& -0\ .011 \sin(-13g'+8g_4) - 0\ .009 \cos(-13g'+8g_4) \\
\delta\pi = & +0\ .35 \sin(-g'+g_4) + 0\ .19 \cos(-g'+g_4)
\end{aligned}
$$

The combinations of $N=2g'$ with $U=-3g'+3g_4$ and $U=-11g'+8g_4$ give rise to terms having the same argument. They have not, however, been computed, because we have not $\varkappa_1=\varkappa_2$, and therefore require $\varkappa_1$ and $\varkappa_2$, and it is easy to assure one's self that they are only of the order of magnitude with those of the same argument just given. In fact $\varkappa_1$, $\varkappa_2$, etc., are 70 times smaller, while K_1, K_2, etc., can not be much more than 70 times greater, and the results are not to be doubled as in the other terms. They can not, therefore, amount to more than one or two hundredths of a second.

The perturbations of θ are much smaller than those of π, and may therefore be neglected entirely.

We have now to consider the terms into which the square of ν enters. By combining the values of K_1, K_2, etc., for $U=0$, which have been already given, with the values of $\varkappa_1$, $\varkappa_2$, etc., in which i is not zero, we shall find a term corresponding to each value of N (pp. 217–224), and proportional to $\varkappa_1$, $\varkappa_2$, etc., the largest of which will be that corresponding to $2\,\varepsilon$, but none of which will amount to one-hundredth of a second. But the corresponding terms, where $i=0$, will be of long period, and therefore, owing to the large values of ν, may become sensible. The largest of these will correspond to $N=2\,\theta$ and $N=2\,\pi$, both of which belong to that class in which

$$\varkappa_1=-\varkappa_2=\varkappa_4;\quad \varkappa_3=\varkappa_5=\varkappa_6=0.$$

Their computation is as follows:

N	$2\,\theta$	$2\,\pi$
U	0	0
$K_1-K_2+K_4$	$+\ 1.85$	$+\ 1.85$
$\varkappa_1$	$+\ .0020$	$+\ .0038$
K'	$+\ .0037$	$+\ .0070$
ν^2	15450	3498
$2\,\varepsilon_s$	$-0.''003$	$+0.''006$

The action of Venus, therefore, gives rise to the terms of long period

$$\delta\,\varepsilon = -0.''003 \sin(2\,\theta - 2\,\pi')$$
$$+0.''006 \sin(2\,\pi - 2\,\pi')$$

We have finally to consider the terms of long period in which neither N nor U are zero. To find these terms we take the table of daily motions of the argument corresponding to each term of x^2, y^2, etc., and compare with the daily motion of the several arguments composed of the mean anomalies of Venus and the Sun, and select those of which the sum or difference nearly vanish. All these combinations which, as it seems to me, can merit examination are given in the following table, which includes also a computation of the probable order of magnitude of the coefficient for the perturbation of mean longitude. The first five columns of this table show the combinations of the several indices i, i', i'', j, j', which make the mean motion of the argument

$$i\,l + i'\,\pi + i''\,\theta + j\,(g_4 - g') + j'\,g'$$

very small.

The sixth column gives the mean daily motion of the argument in seconds. The first line, for example, shows that twice the mean motion of the Moon's node, plus five times the mean motion of the Earth, minus three times that of Venus, is only 56″, and that the corresponding inequality is therefore one of very long period.

The seventh column gives the value of ν, which really represents the period of the argument in sidereal revolutions of the Moon. Multiplying ν by .075, we shall have the period of the inequality in years.

The three next columns give, as in the case of Mercury, the probable order of magnitude of the logarithms of ν^2, $\varkappa_i$, and K_i, the last being deduced by induction from the law of development of $\frac{1}{\rho^3}$.

The last two columns give log ε_c and the probable limiting order of magnitude of ε_c itself, hence concluded. These quantities may also be regarded as the probable values of ε_s.

It will be seen that several terms have been included, the components of which are not found in the preceding developments. This has been done to include every term which could possibly become sensible in the course of a few centuries. The magnitudes assigned to $\varkappa_i$ and K_i in these cases are hardly more than rough guesses, and it must be understood that precision is not aimed at in any case.

i	i'	i''	j	j'	ω	ν	$lg\,\nu^2$	$lg\,\varkappa_1$	$lg\,K_1$	$lg\,\varepsilon_c$ +10	ε_c
					″						″
0	0	2	−3	+2	+56	847	5.9	−2.7	−1.5	. . . .	.00
0	2	0	−2	+1	−89	533	5.6	−2.4	−0.8	. . . .	.00
1	−3	0	−24	+2	+61	780	5.8	−4.6	−3.0	7.8	.00
1	−3	0	−16	−3	+76	624	5.6	−4.6	−2.7	7.5	.00
1	−3	−2	−21	0	+4	12000	8.2	−6. (?)	−1.4	8.4	.02
1	−1	0	−26	+3	−28	1695	6.5	−1.3	−4.0	8.8	.07
1	−1	0	−18	−2	−13	3560	7.1	−1.3	−2.0	11.7	25.
1	−1	0	−10	−7	+1.6	29600	9.0	−1.3	−5.0	10.3	2.
1	−1	−2	−23	+1	−84	560	5.5	−4.0	−2.	7.1	.00
1	−1	2	−21	0	+43	1100	6.1	−3.6	−1.5	8.6	.04
1	1	0	−20	−1	−102	465	5.3	−1.1	−2.5	9.4	.2
1	0	0	−23	+1	−65	730	5.7	−3.6	−2.0	7.7	.00
1	0	2	−18	−2	+6	8000	7.8	−5. (?)	−2.7	7.7	.005
2	0	0	−38	−3	−116	409	5.2	−0.3	−5.0	7.8	.01
2	0	2	−41	−1	−59	802	5.8	−3.9	−4.5	5.3	.00
2	−2	0	−44	+1	−41	1150	6.1	−3.1	−5.0	5.9	.00
2	−2	2	−55	+8	+0.7	67000	9.6	−5. (?)	−7.0	5.5	.00
0	0	1	2	−1	+700	68	3.7	−1.1	−0.5	7.3	.00
0	0	1	−3	+2	+247	192	4.6	−1.1	−0.5	8.2	.02
0	2	−1	−2	+1	+102	465	5.3	−3.3	−0.5	7.5	.00
1	1	−1	−20	−1	+88	540	5.5	−1.8	−2.5	8.8	.06
1	3	−1	−22	0	+0.4	120000	10.2	−6. (?)	−1.5	10.3	2.
0	2	2	3	−2	18	2630	6.8	−5. (?)	−0.5	8.9	.08

The only coefficients which from their magnitude merit farther examination are those corresponding to terms 7, 8, 9, 11, 12, 22, and 23. Of these numbers 9 and 22 are of such immensely long period, several thousand years, that their effect could not possibly be sensible in all the observations hitherto made on the Moon. The first can hardly be accurately computed by any method hitherto employed in celestial mechanics, because the secular variations of the perihelia and nodes of the Earth and Venus will sensibly affect the arguments. In fact, the rigorous development does not contain simply the term $3g' - 10g_4$, but $3l' - 10l_4$ plus many combinations of angles varying in consequence of the secular variations of the planetary orbits. In the present theory these angles are supposed constant, the theory is therefore not rigorous for very long intervals. The further consideration of these two terms will therefore be omitted entirely.

An examination of 7 shows that its real value is smaller than that here assigned, so that it also may be omitted. We therefore begin with the exact computation of the eighth term, which is HANSEN's first inequality. An examination of the values of K_1, K_2, etc., shows that there are really six terms in N, each of which, being combined with some value of U, will make

$$N \pm U = \varepsilon - \pi + 16g' - 18g_4$$

These terms, together with the numerical values of K_1, K_2, etc., are investigated as follows:

We find that the analytical expressions for x^2, y^2, z^2, $y\,z$, $z\,x$, and $x\,y$ contain the following terms, which, combined with the proper multiples of the mean anomalies of the Earth and Venus, will give rise to terms having the argument $\varepsilon-\pi+16\,g'-18\,g_4$

$$\frac{x^2}{a^2}=\left\{-e+\frac{1}{8}e^3+2\,e\,\gamma^2+\frac{13}{12}e\,m^2\right\}\cos(l-\pi)$$

$$-\frac{21}{8}e\,e'\,m\cos(l-\pi-g')$$

$$+\frac{87}{16}e\,e'\,m\cos(l-\pi+g')$$

$$+\left\{-\frac{45}{16}e\,m-\frac{773}{64}e\,m^2\right\}\cos(l-\pi+2\,g')$$

$$+\frac{55}{32}e\,m^2\cos(l-\pi-2\,g')$$

$$-\frac{105}{16}e\,e'\,m\cos(l-\pi+3\,g')$$

$$\frac{y^2}{a^1}=\left\{-e+\frac{1}{8}e^3+2\,e\,\gamma^2+\frac{13}{12}e\,m^2\right\}\cos(l-\pi)$$

$$-\frac{21}{8}e\,e'\,m\cos(l-\pi-g')$$

$$-\frac{3}{16}e\,e'\,m\cos(l-\pi+g')$$

$$+\left\{\frac{45}{16}e\,m+\frac{773}{64}e\,m^2\right\}\cos(l-\pi+2\,g')$$

$$-\frac{55}{32}e\,m^2\cos(l-\pi-2\,g)$$

$$+\frac{105}{16}e\,e'\,m\,(l-\pi+3\,g')$$

$$\frac{x\,y}{a^2}=\left\{-\frac{45}{16}e\,m-\frac{773}{64}e\,m^2\right\}\sin(l-\pi+2\,g')$$

$$-\frac{55}{32}e\,m^2\sin(l-\pi-2\,g')$$

$$+\frac{45}{16}e\,e'\,m\sin(l-\pi+g)$$

$$-\frac{105}{16}e\,e'\,m\sin(l-\pi+3\,g)$$

$$z^2=-4\,e\,\gamma^2\cos(l-\pi)$$

Reducing to numbers, these values become

$$
\begin{aligned}
\frac{x^2}{a^2} = &-.05433\cos(\varepsilon-\pi)\\
&-.00018\cos(\varepsilon-\pi-g')\\
&+.00037\cos(\varepsilon-\pi+g')\\
&-.01525\cos(\varepsilon-\pi+2g')\\
&+.00053\cos(\varepsilon-\pi-2g')\\
&-.00045\cos(\varepsilon-\pi+3g')
\end{aligned}
\qquad
\begin{aligned}
\frac{y^2}{a^2} = &-.05433\cos(\varepsilon-\pi)\\
&-.00018\cos(\varepsilon-\pi-g')\\
&-.00001\cos(\varepsilon-\pi+g')\\
&+.01525\cos(\varepsilon-\pi+2g')\\
&-.00053\cos(\varepsilon-\pi-2g')\\
&+.00045\cos(\varepsilon-\pi+3g')
\end{aligned}
$$

$$
\begin{aligned}
\frac{xy}{a^2} = &-.01525\sin(\varepsilon-\pi+2g')\\
&-.00053\sin(\varepsilon-\pi-2g')\\
&+.00020\sin(\varepsilon-\pi+g')\\
&-.00045\sin(\varepsilon-\pi+3g')
\end{aligned}
\qquad
z^2 = -.00044\cos(\varepsilon-\pi)
$$

We thus have the following values of $\varkappa_i$ for the six arguments N:

N	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	U
$\varepsilon-\pi-2g'$	$+.00053$	$-.00053$	0	$-.00053$	$-18g'+18g_4$
$\varepsilon-\pi-g'$	$-.00018$	$-.00018$	0	0	$-17g'+18g_4$
$-\pi$	$-.05433$	$-.05433$	$-.00044$	0	$-16g'+18g_4$
$\varepsilon-\pi+g'$	$+.00037$	$-.00001$	0	$+.00019$	$-15g'+18g_4$
$\varepsilon-\pi+2g'$	$-.01525$	$+.01525$	0	$-.01525$	$-14g'+18g_4$
$\varepsilon-\pi+3g'$	$-.00045$	$+.00045$	0	$-.00045$	$-13g'+18g_4$

From the numerical developments already given we obtain the following values of K_1, K_2, etc., the last value of U being omitted owing to the necessary minuteness of the term to which it gives rise:

U	32 K_1	32 K_2	32 K_3	32 Δ	32 K_4
$-18g'+18g_4$	-0.8052	-0.3925	$+1.2021$	$+.0044$	$+0.2972$
$-17\ \ +18$	. . . . .	. . . . . .	-0.0144	. . . . . .	. . . . . .
$-16\ \ +18$	-0.1095	-0.0253	$+0.12068$	$-.0141$	. . . . . .
$-15\ \ +18$	-0.0039	$+0.0008$	$+0.0027$	$-.0004$	-0.0201
$-14\ \ +18$	-0.011	$+0.006$	. . . . . .	$-.005$	-0.013
U	32 K'_1	32 K'_2	32 K'_3	32 Δ	32 K'_4
$-18g'+18g_4$	-0.0067	-0.2821	$+0.2864$	$-.0024$	$+0.3878$
$-17\ \ +18$	. . . . . .	. . . . . .	$+0.0055$	. . . . . .	. . . . . .
$-16\ \ +18$	$+0.0246$	$+0.0599$	-0.08774	$-.0032$	. . . . . .
$-15\ \ +18$	-0.0093	$+0.0096$	$+0.0013$	$+.0016$	-0.0107
$-14\ \ +18$	$+0.008$	-0.007	-0.004	$-.003$	-0.017

We have found that the sum $K_1+K_2+K_3$ ought to vanish. The column 32Δ shows the sum of the computed numerical values, and hence, the sum of the errors with which the computations are affected. As has already been remarked, the difficulty of computing accurate values of K_1 and K_2 by the method of computation adopted is very great, because these quantities appear as small differences of very large numbers, while the same difficulty does not attend the computation of K_3. Hence, in the case of the most important term, that depending on the argument $-16g'+18g_4$, we may conclude with considerable probability that the outstanding errors are entirely in the quantities K_1 and K_2. The only other error of the least importance is that corresponding to the argument $-14g'+18g_4$. Here the value of Δ is so great that we may conclude that the values of K_1, K_2, and K_3 are uncertain to almost their entire amount, and the chances are decidedly in favor of the hypothesis that their values are numerically too great.

Combining the terms by the formulæ (111) or (112) we find that the different combinations give rise to the following values of 32 K and 32 K′:

N	U	32 K	32 K′
$\varepsilon-\pi-2g'$;	$-18g'+18g_4$	$-.000305$	$-.000424$
$\varepsilon-\pi-g'$;	$-17\quad+18$	$-.000001$	$-.000003$
$\varepsilon-\pi$;	$-16\quad+18$	$+.004728$	$+.006503$
$\varepsilon-\pi+g'$;	$-15\quad+18$	0	$-.000003$
$\varepsilon-\pi+2g'$;	$-14\quad+18$	$+.000427$(?)	$+.000518$(?)
Sum		$+.004849$	$+.006591$

For all these values of N we have

$$i=+1$$

$$i'=-1$$

$$i''=0$$

which, being substituted in the values of ε_c and ε_s, give

$$\varepsilon_c= 0.''004226\,\nu^2\,K+0.''00556\,\nu\left(K+\tfrac{1}{2}\frac{dK}{dw}\right)$$

$$\varepsilon_s=-0.''004226\,\nu^2\,K'-0.''00556\,\nu\left(K'+\tfrac{1}{2}\frac{dK'}{dw}\right)$$

The annual motion of the argument

$$N-U=\varepsilon-\pi+16g'-18g_4$$

taking for the motions of g' and g_4 the mean sidereal motions of the Earth and Venus is

$$-4750.''$$

which gives

$$\nu=3648$$

We thus have

$$\varepsilon_c=+8.''52$$

$$\varepsilon_s=-11.58$$

or

$$\begin{aligned}\delta\varepsilon &= -11.''58 \sin(\varepsilon-\pi+16g'-18g_4)\\ &\quad + 8.52\cos(\varepsilon-\pi+16g'-18g_4)\\ &= 14.38\sin(18g_4-16g'-g+36^\circ 20')\end{aligned}$$

The adopted mass of Venus is about $\frac{1}{20}$ less than that used by HANSEN and DELAUNAY. In increasing the result by $\frac{1}{20}$ the coefficient will agree very nearly with that adopted by HANSEN in his *Tables de la Lune*, but will still be more than 1″ smaller than that found by DELAUNAY.

It will be seen that had we omitted the doubtful term in K and K′, the coefficient would have been one-twelfth, or about 1″ smaller, and it is not improbable that this term is numerically too large. If it were diminished the final value of the coefficient would diverge yet more widely from the values of HANSEN and of DELAUNAY.

It is a curious fact that when we consider terms depending on these elevated multiples of the mean longitudes, the terms of the second order become proportionally much larger than in the case of the terms usually considered in problems of celestial mechanics. In computing the term under consideration we have determined the direct action of the planet on the Moon, on the supposition that both the Earth and Venus move in their elliptic orbits of 1800. Since the entire action of Venus on the Moon is a quantity barely sensible, it might be considered that the change in this action due to the perturbations of a few seconds which the relative coordinates of these bodies suffer in consequence of their mutual action would necessarily be minute in the extreme, and entirely unworthy of consideration. But it must be remembered that any individual term of high order of the class under consideration really represents only a very minute portion of the total disturbing force of the planet—a minute residuum, in fact, which could produce no appreciable effect but for its period very nearly coinciding with that of some argument in the motion of the Moon. Now, we must not conclude without examination that the effect of a term in the perturbations would be insensible alongside a term of this character.

Let us inquire to what extent the values of K and K′ may be altered by the perturbations. And, first, let us consider the changes introduced by the perturbations of ρ^3. The value of ρ is determined by the equation

$$\rho^2 = r'^2 + r_4^2 - 2\, rr' (\cos v_4 \cos v' + \cos \gamma \sin v_4 \sin v')$$

v_4 and v' being the distances of Venus and of the Earth from the common node of their orbits, and γ the mutual inclination of the orbits. For our present purpose we may put $\cos \gamma = 1$, and may suppose the perturbations of v and v' to be the same as those of the longitudes in the orbits. These suppositions give, by putting $v_4 - v' = \mathrm{V}$,

$$\rho^2 = r'^2 + r_4^2 - 2\, r' r_4 \cos \mathrm{V}$$

and

$$\delta \frac{1}{\rho_3} = \frac{3 (r_4 \cos \mathrm{V} - r')}{\rho^5} \delta r' + \frac{3 (r' \cos \mathrm{V} - r_4)}{\rho^5} \delta r_4 - \frac{3\, r' r_4}{\rho^5} \sin \mathrm{V}\, \delta \mathrm{V}$$

According to **Le Verrier** (*Annales del Observatoire de Paris*, *Tomes* IV, VI) the largest terms in $\delta r'$, δr_4, $\delta v'$, and δv_4 depending on the mutual action of Venus and the Earth are

$$\begin{aligned}
\delta r' = & - 1.''1 \cos\ (l_4 - l') \\
& + 3.''3 \cos 2\, (l_4 - l') \\
& + 0.''5 \cos 3\, (l_4 - l')
\end{aligned}$$

$$\begin{aligned}
\delta r_4 = & 0.''7 \cos\ (l_4 - l') \\
& + 3.''1 \cos 2\, (l_4 - l') \\
& - 2.''6 \cos 3\, (l_4 - l')
\end{aligned}$$

$$\begin{aligned}
\delta v' = & + 4.''9 \sin\ (l_4 - l') \\
& - 5.''6 \sin 2\, (l_4 - l') \\
& - 0.''7 \sin 3\, (l_4 - l') \\
& + 2.''5 \cos (-2\, l_4 + 3\, l') \\
& + 1.''6 \cos (-3\, l_4 + 4\, l') \\
& + 1.''0 \sin (-3\, l_4 + 5\, l') \\
& + 0.''3 \cos (-3\, l_4 + 5\, l')
\end{aligned}$$

$$\begin{aligned}
\delta v_4 = & - 4.''5 \sin\ (l_4 - l') \\
& - 10.''4 \sin 2\, (l_4 - l') \\
& + 6.''6 \sin 3\, (l_4 - l') \\
& - 3.''2 \cos (-2\, l_4 + 3\, l') \\
& 0.''6 \cos (-3\, l_4 + 4\, l') \\
& - 1.''5 \cos (-4\, l_4 + 5\, l') \\
& - 1.''3 \sin (-3\, l_4 + 5\, l') \\
& - 0.''4 \cos (-3\, l_4 + 5\, l')
\end{aligned}$$

Whence

$$
\begin{aligned}
\delta V = & -9.''4 \sin \ (l_4 - l') \\
& -4.''8 \sin 2\,(l_4 - l') \\
& +7.''3 \sin 3\,(l_4 - l') \\
& -5.''7 \cos (-2\,l_4 + 3\,l') \\
& -2.''2 \cos (-3\,l_4 + 4\,l') \\
& -1.''5 \cos (-4\,l_4 + 5\,l') \\
& -2.''3 \sin (-3\,l_4 + 5\,l') \\
& -0.''7 \cos (-3\,l_4 + 5\,l')
\end{aligned}
$$

These expressions are now to be substituted in the above expressions for $\delta \frac{1}{\rho^3}$. In making this substitution we may, in a first approximation, put

$$
\begin{aligned}
r' &= a' = 1 \\
r_4 &= a_4 = 0.72 \\
V &= l_4 - l'
\end{aligned}
$$

and for brevity, we shall put

$$l_4 - l' = L$$

We thus find

$$
\begin{aligned}
\rho^5\, \delta \frac{1}{\rho^3} = & +10.''0 \\
& +15.''2 \cos \ L \\
& -38.''3 \cos 2\,L \\
& +\ 7.''1 \cos 3\,L \\
& -\ 2.''5 \cos (4\,L - 2\,l_4) \\
& +\ 2.''5 \cos (6\,L - 2\,l_4) \\
& -\ 6.''2 \sin (2\,L - \ l_4) \\
& -\ 2.''4 \sin (3\,L - \ l_4) \\
& +\ 4.''6 \sin (4\,L - \ l_4) \\
& +\ 2.''4 \sin (5\,L - \ l_4) \\
& +\ 1.''6 \sin (6\,L - \ l_4) \\
& -\ 0.''8 \sin (4\,L - 2\,l_4) \\
& +\ 0.''8 \sin (6\,L - 2\,l_4)
\end{aligned}
$$

We must now substitute for L and l_4 their values in g' and g_4, namely,

$$
\begin{aligned}
L &= -g' + g_4 + 29.^\circ 1 \\
l_4 &= \quad\ \ g_4 + 129.^\circ 45
\end{aligned}
$$

when the expression in question becomes in units of the seventh place of decimals.

g'	g_4	$\rho_5\,\delta\,\frac{1}{\rho_3}$	
		cos	sin
0	0	+485	......
−1	1	+645	− 359
−2	1	+286	− 97
−2	2	−980	+1576
−3	2	+ 78	− 82
−3	3	+ 20	− 344
−4	2	+121	− 44
−4	3	− 53	+ 218
−5	4	+ 29	+ 116
−6	4	− 29	+ 126
−6	5	+ 53	+ 53

If we multiply this expression by that for $\frac{24}{\rho^5}$ given in pp. 252–257, and retain only the terms which depend on the argument $-16g'+18g_4$, we find the sum of those terms to be

Sum of positive products . . . +.001 28 cos+.001 12 sin

Sum of negative products . . . −.001 29 cos−.001 15 sin

$$\text{Term of } \delta\,\frac{24}{\rho^3} = -.000\,01 \cos(-16\,g'+18\,g_4) - .000\,03 \sin(-16\,g'+18\,g_4)$$

It appears that although the individual products are large enough sensibly to alter the expression for $\frac{1}{\rho^3}$, yet the sums of the different products so destroy each other that the entire term is quite insensible. This, however, may not be true of the values of $\frac{(x_4+x')^2}{\rho^5}$, etc. The computation of the inequalities in these quantities can not, however, be considered in the present paper.

A closer examination of all the other terms (Tabulated on p. 282) shows that their true values are generally smaller than those here assigned them. Their more exact computation may therefore be dispensed with.

§ 22.

ACTION OF MARS.

The adopted mass of this planet being $\frac{1}{3\,000\,000}$ we find by proceeding as in the case of Venus,

$$\mu_4\,a^3 = 0.''000\,384\,5$$

This value being substituted in the equations (114) the secular terms become

$$\delta\,\varepsilon = 0.''000\,3845\left\{-2.023\,K'_0 - 1.011\,\frac{d\,K'_0}{d\,x} + .0005\,\frac{d\,K'_0}{e\,d\,e}\right\}n\,t$$

$$\delta\,\pi = 0.''000\,3845\left\{+.0293\,K'_0 + .0147\,\frac{d\,K'_0}{d\,x} + \;.526\,\frac{d\,K'_0}{e\,d\,e}\right\}n\,t$$

$$\delta\,\theta = 0.''000\,3845\left\{-.0074\,K'_0 - .0037\,\frac{d\,K'_0}{d\,x} + \;.125\,\frac{d\,K'_0}{\gamma\,d\,\gamma}\right\}n\,t$$

Among the terms of small multiples those multiplied by z_4^2 are comparatively small. If we omit them we shall have in the equations (103) for $U=0$

$$32\;K_3 = -17.3$$

K_3 being, when we neglect z_4, the negative of one-half the constant term of $\frac{1}{\rho^3}$. Using the values of $\varkappa_3$ and $\varkappa_1$ found in the case of Venus, we find

$$32\;K'_0 = +\;8.6$$

$$32\,\frac{d\,K'_0}{e\,d\,e} = +\,26.4$$

Substituting in the above equations, we find, by putting $n=84$, as in the case of Venus,

$$\delta\,\varepsilon = -0.''035\,t$$

$$\delta\,\pi = +0.''028\,t$$

$$\delta\,\theta = -0.''008\,t$$

The equations (113) become by substituting the preceding value of $\mu_4\,a^3$

$$\varepsilon_c = (\;\;.''000\,583\,i - .''000\,008\,i' + .''000\,002\,i'')\,\nu^2\,K$$

$$+\quad .''000\,778\,\nu\,K + .''000\,389\,\nu\,\frac{d\,K}{d\,x}$$

$$\varepsilon_s = (-\,''000\,583\,i + .''000\,008\,i' - .''00\,002\,i'')\,\nu^2\,K'$$

$$-.''000\,788\,\nu\,K - .''000\,389\,\nu\,\frac{d\,K}{d\,x}$$

The corresponding coefficients for the perigee and node are too small to be worth computing.

Proceeding as in the case of Venus we begin with those terms in which i, i', and and i'' are all zero, so that the term contains the mean longitudes of only the Earth and Mars. In the case of these terms we have

$$K = \mp 0.495\ K'_3$$

$$K' = -0.495\ K_3$$

In the terms of small multiples we may take as an approximate value of K_3 the corresponding coefficient in the development of $-\frac{1}{2}\frac{1}{\rho^3}$.

We now select the following terms from the development of $\frac{1}{\rho^3}$ in the case of Mars and the Earth as the only ones in which either from the length of the period or the magnitude of the coefficient we may find sensible values of ε_c or ε_s.

U	K_3	K'_3	2 K	2 K′	ν	$2\varepsilon_c$	$2\varepsilon_s$
						″	″
$-g_4$	$-.340$	$-.034$	$\pm .034$	$+.337$	∓ 25.14	$-.001$	$+.007$
$-g_4+g'$	$+.553$	$+.728$	$\mp .721$	$-.547$	± 28.55	$-.014$	$+.011$
$-2g_4+g'$	$+.209$	$+.342$	$\mp .338$	$-.207$	∓ 210.9	$+.055$	$-.033$
$-2g_4+2g'$	$+.189$	$-.675$	$\pm .668$	$-.187$	± 14.28	$+.007$	$+.002$
$-3g_4+2g'$	$+.149$	$-.374$	$\pm .370$	$-.148$	± 33.02	$+.009$	$+.004$
$-3g_4+3g'$	$-.474$	$+.186$	$\mp .184$	$+.469$	± 9.52	$-.001$	$-.003$
$-4g_4+2g'$	$+.068$	$-.137$	$\pm .136$	$-.067$	∓ 105.5	$-.010$	$-.001$
$-5g_4+3g'$	$-.160$	$+.030$	$\mp .030$	$+.158$	± 39.14	$-.001$	$-.005$
$-6g_4+3g'$	$-.058$	$+.006$	$\mp .006$	$+.057$	∓ 70.30	0	$+.003$

The inequalities in question are therefore

$$-.''001 \cos g_4 - .''007 \sin g_4$$

$$-.''014 \cos (g_4 - g') - .''011 \sin (g_4 - g')$$

etc. etc.

We next consider the terms into the arguments of which the elements of the Moon's orbit enters. Proceeding as in the case of Venus we find the following terms of long period. The computation of the probable order of magnitude of the inequalities is conducted in the same way as in the case of Venus.

i	i'	i''	j	j_4	ω	$\log \nu^2$	$\log \varkappa_1$	$\log K_1$	$\log \varepsilon$ +10	ε
					$''$					$''$
0	0	2	− 2	+ 4	68	5.7	−2.7	−0.4	7.2	0.02
0	2	0	− 5	+ 9	42	6.1	−2 4	−2.0	6.9	.01
1	−3	0	−29	+30	69	5.7	−4.6	−3.2	4.7	.00
1	−3	0	−21	+15	18	6.8	−4.6	−3.4	5.6	.00
1	−1	0	−26	+24	57	5.8	−1.3	−3.0	8.3	.02
1	0	0	−32	+35	78	5.6	−3.6	−3.3	5.5	.00
1	1	0	−31	+33	96	5.4	−1.1	−3.3	-.0	.01
1	−1	2	−28	+28	125	5.2	−3.6	−3.0	5.4	.00
1	−1	−2	−24	+20	10	7.4	−3.6	−3.0	7.6	.00
1	1	−2	−29	+29	29	6.4	−4.3	−3.1	5.9	.00
0	0	1	− 1	+ 2	34	6.2	−1.1	−0.5	9.0	0.10
0	2	−1	− 4	+ 7	6	7.8	−3.3	−1.0	8.7	.07
1	−1	1	−27	+26	91	5.4	−2.3	−3.2	6.7	.00
1	−1	1	−35	+41	6	7.8	−2.3	−4.0	8.3	.02
1	−1	−1	−25	+22	23	6.6	−2.3	−3.5	7.6	.00
1	1	−1	−30	+31	64	5.7	−1.8	−3.2	7.4	.00
1	1	1	−23	+18	7	7.7	−3.6	−3.0	7.9	.01

The largest of these terms, the eleventh, is one in which it is reasonably certain that the value of K_i does not exceed the limit here assigned, because it contains the latitude of Mars as a factor. A coefficient of 0.″1 could scarcely be detected in observations; the computation of the term is therefore dispensed with.

§ 24.

ACTION OF JUPITER.

With the adopted mass of Jupiter (BESSEL'S) we find

$$\mu_4 a^3 = 1''101$$

The substitution of this value in the expressions (112) and (114) give for the secular and periodic terms in the perturbations

$$\delta\varepsilon = \left\{-2.''227\,K'_0 - 1.''113\,\frac{dK'_0}{dx}\right\}nt$$

$$\delta\pi = \left\{\;0.''032\,K'_0 + 0.''016\,\frac{dK'_0}{dx} + 0.''579\,\frac{dK'_0}{e\,de}\right\}nt$$

$$\delta\theta = \left\{-0''.008\,K'_0 - 0.''004\,\frac{dK'_0}{dx} + 0.''138\,\frac{dK'_0}{\gamma\,d\gamma}\right\}nt$$

$$\varepsilon_c = (1.''67\;\;i - 0.''024\,i' + 0.''006\,i'')\,\nu^2 K$$

$$+ 2.''227\,\nu K + 1.''113\,\nu\frac{dK}{dx}$$

$$\varepsilon_s = (-1.''67\;\;i + 0.''024\,i' - 0.''006\,i'')\,\nu^2 K'$$

$$-2''227\,\nu K' - 1.''113\;2\;\nu\frac{dK'}{dx}$$

$$\pi_e = -0.''579\,\nu\frac{d\,\mathrm{K}}{e\,d\,e}$$

$$\pi_s = 0.\;579\,\nu\frac{d\,\mathrm{K}'}{e\,d\,e}$$

$$\theta_e = -0.''138\frac{d\,\mathrm{K}}{\gamma\,d\,\gamma}$$

$$\theta_s = 0.\;138\frac{d\,\mathrm{K}'}{\gamma\,d\,\gamma}$$

Owing to the distance of Jupiter the developments (103) are comparatively simple, and it is unnecessary to enter extensively into their numerical details. If we put

r_4, the radius vector of Jupiter.

V, the angular distance between Jupiter and the Earth, as seen from the Sun.

L, the difference of their mean longitudes,

we have

$$\rho^2 = r_4^2\left(1 - 2\frac{r'}{r_4}\cos\mathrm{V} + \frac{r'^2}{r_4^2}\right)$$

$$\frac{1}{\rho^3} = \frac{1}{r_4^3}\left\{1 + 3\frac{r'}{r_4}\cos\mathrm{V} + \frac{15}{2}\frac{r'^2}{r_4^2}\cos^2\mathrm{V} - \frac{3}{2}\frac{r'^2}{r_4^2} + \text{etc.}\right\}$$

$$\frac{1}{r_4^3} = \frac{1}{a_4^3}\left\{1 + \frac{3}{2}e_4^2 + 3\,e_4\cos g_4 + \frac{9}{2}e^2_4\cos 2\,g_4 + \text{etc.}\right\}$$

Beginning with the constant term of $\frac{1}{\rho^3}$ we find that its value is, with sufficient approximation,

$$\frac{1}{a_4^3}\left(b_{\frac{3}{2}}^{(0)} + \frac{3}{2}e_2\right)$$

$b_{\frac{3}{2}}^{(0)}$ being the constant term in the development of

$$(1 - 2\,\alpha\cos\mathrm{L} + \alpha^2)^{-\frac{3}{2}}$$

The value of this constant is

$$b_{\frac{3}{2}}^{(0)} = 1.0883.$$

In the expressions (103) the terms of $\frac{z_4^2}{\rho^5}$ will be very small in comparison with those of $\frac{1}{\rho^3}$, we have therefore only to consider the development of the latter term in forming the values of K_3 and K'_3. Beginning with the constant term, we have

$$K_3 = -.003\,876 \qquad K'_3 = 0$$

From this we have

$$K'_0 \quad = +.001\,920$$

$$\frac{d\,K'_0}{d\,w} = K'_0 = +.000\,008$$

$$\frac{d\,K'_0}{e\,d\,e} = +.\ 00\,589$$

$$\frac{d\,K'_0}{\gamma\,d\,\gamma} = -.\ 00\,774$$

Proceeding as in the case of Venus we find, for the secular terms:

$$\delta\,l = -0.''720\ t$$

$$\delta\,\pi = +0.''573\ t$$

$$\delta\,\theta = -0.''179\ t$$

We have next to consider the periodic terms arising from the constant of $\frac{1}{\rho^3}$, which can readily be shown to be too small to be taken into account. In a first approximation to the corresponding values of K_i we may neglect x' and y' alongside of x_4 and y_4. We thus find

$$K_1 = +.0019$$

$$K_2 = +.0019$$

$$K_3 = -.0039$$

$$K_4 = 0$$

Referring to the tables, p. 217, we find that where i, or the coefficient of (l) is not zero, the largest value of $\varkappa$ is less than 0.5, so that the largest value of K or K' is less than .001. In these terms the coefficient ν will not differ much from unity, the values of ε_c and ε_s will therefore never exceed 0.''002.

In the seven terms of the table referred to which follow the first, i vanishes, and ν is therefore large. But $\varkappa$ and the coefficient of i' and i'' are in these cases very small, and the largest terms in N are of the second class, in which $K_1 + K_2 = 0$; $K_3 = 0$, so that the principal term depends on $K_4\ \varkappa_4$. The constant term of K_4, which we have above put equal to zero, must be less than .0006, while the largest value of k_4 is less than .003. K or K' must therefore be less than .000002, while ν is between 100 and 150. Substituting these values in ε_c and ε_s we find that the latter do not attain a sensible value. The constant term of $\frac{1}{\rho^3}$ therefore gives rise to no sensible periodic inequality.

Passing now to the periodic terms we shall begin with the combination of these terms with the zero, or first term of N. This term is of the first class (150), p. 275, so that we only want the values of K_3 and, as already shown, may neglect the square of z_4, and so consider only the development of $\frac{1}{\rho^3}$. The largest periodic terms in this development are numerically, putting L for the difference of mean longitude of Jupiter and the Earth,

$$\frac{1}{\rho^3} = .004\,38 \cos L + .001\,08 \cos 2\,L + .001\,03 \cos g_4$$

We have therefore the following combinations of the preceding values of U with the zero term in N:

U	K_3	K'_3	ν	ε_s
L	$-.00\,219$	$\pm.00\,100$	14.6	$\mp.''032$
2 L	$-.00\,054$	$\pm.00\,024$	7.4	$\mp.''004$
g_4	$-.00\,052$	$\pm.00\,023$	$+158.7$	$\mp.''080$

The resulting inequalities in the mean longitude are

$$\begin{aligned}\delta\,\varepsilon &= 0.''064 \sin\ (l_4 - l') \\ &\quad + 0.''008 \sin 2\,(l_4 - l') \\ &\quad - 0.''060 \sin g_4\end{aligned}$$

The combination of the terms in $\frac{1}{\rho^3}$ which depend on multiples of the mean anomaly of Jupiter with those which depend on the lunar perigee and node lead to terms of long period, but none of the coefficients are sensible. The only terms of U which, combined with those terms of N containing ε can be of long period, depend on such high powers both of α and of e_4 and e' that they can not be sensible.

Postscript (added August, 1894).—In the above reasoning it is not noticed that in the developments (103) the periodic term having much the largest coefficient depends on the argument twice the mean longitude of Jupiter. This term gives rise to the "Jovian evection," discovered by Mr. NEISON, and computed with great precision by Mr. HILL, in Vol. III of the present series.

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